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Pseudorandom number generators as applied in reliability analysis

Streszczenie: *Układ elektroenergetyczny uszkadza się głównie na skutek czynników o charakterze losowym. W takim przypadku bardzo pomocne są komputerowe metody pozwalające na symulacje zjawisk zachodzących w systemie. Metody te wraz z rozwojem technik komputerowych stały się coraz częściej używanym narzędziem badawczym. Do prawidłowego zamodelowania działania elementów systemu elektroenergetycznego niezbędne jest odwzorowanie rozkładów funkcji niezawodności oraz funkcji odnowy poszczególnych elementów, Rozkłady prawdopodobieństwa czasu poprawnej pracy lub czasu trwania awarii elementów składowych można zamodelować przy pomocy wybranych generatorów liczb pseudolosowych. W artykule zaprezentowane i przetestowane zostaną generatory podstawowych, najczęściej wykorzystywanych w analizach niezawodnościowych układów elektroenergetycznych, rozkładów (rozkładu normalnego, rozkładu Weibulla, rozkładu wykładniczego i rozkładu logarytmiczno - normalnego).*

Abstract: *An electric power system is damaged mainly due to random factors. In this case, computer methods that allow simulation of phenomena occurring in the system are very helpful. These methods, along with the development of computer techniques, have become an increasingly used research tool. In order to properly model the operation of the electric power system elements, it is necessary to map the distributions of the reliability function and the renewal function of individual elements. The probability distributions of the correct operation time or the failure duration of the components can be modelled using selected pseudorandom number generators. The article presents and tests the distribution generators most often used in reliability analyses of electric power systems, and which are considered the standard - normal distribution, Weibull distribution, exponential distribution and log-normal distribution.* ***(Generatory liczb pseudolosowych stosowane w analizie niezawodnościowej układów elektroenergetycznych).***

Słowa kluczowe: generatory liczb pseudolosowych, metody komputerowe, dystrybuanta.
Keywords: pseudo-random number generators, computer methods, cumulative distribution function.

Introduction

The available (universal and Polish) literature describes the simplest random number generators – exponential and normal distribution generator. In reliability analysis, when evaluating the renewal function or the failure duration function, often Weibull distributions or log-normal distribution are encountered. Today, computer simulations are very often applied to analyse the operation and reliability of electric power systems. The use of pseudorandom number generators allows modelling the operation of individual devices. However, this cannot be done properly without knowing the computations for other distributions.

Generating pseudorandom numbers with an appropriate distribution is possible by many methods, differing in ways of approximation or time of finding the appropriate random number. However, a true random number cannot be obtained as a result of software-generated random numbers. This is only possible through a physical generator.

The generators used in the software are based on a uniform distribution generator from 0 to 1. With the uniform distribution generator, the results obtained are converted into numbers with a required random distribution. With a suitable approximation, it can be assumed that the numbers obtained are random numbers with a required distribution, although in fact, the numbers are pseudorandom numbers and the distributions by which they are generated do not fully reflect the nature of the interval, especially at its ends. [1, 2]

This article presents and tests the generators most frequently used to describe the reliability of electric power systems. These include exponential distribution, Weibull distribution, normal distribution and log-normal distribution. [3]

Characteristics of the developed pseudorandom number generators

The authors of the publication have created a computer application that models the generation of pseudorandom numbers according to a given distribution with specific parameters. Below is a description of the mathematical relations to be entered into the simulation software.

For the three tested distributions, the cumulative distribution function was used.

Suppose that a random variable *R* is a uniformly distributed variable over the interval [0, 1]. Let *F* be a continuous and strictly different distribution function of some probability distribution. We define it as:

(1) $$\mathrm{X} = F^{-1}(R)$$

Exponential distribution generator

The random variable W has an exponential distribution if its distribution is expressed by the formula [4]:

2)
$$\mathrm{P}\{\mathrm{W} \le \mathrm{w}\} = F(w) = \begin{cases} 1 - \exp^{-cw} & \text{for } w \ge 0 \\ 0 & \text{for } w < 0 \end{cases}$$

Where $c > 0$ is a certain constant ($1/c$ is the expected value of a random variable *W*). Let [4]:

(3) $$X = c \cdot W$$

The random variable *X* therefore has an exponential distribution with an expected value of one. Because from a variable of exponential distribution with expected value $c = 1$ it is possible to obtain a random variable with exponential distribution with any value of parameter *c*, for further calculations, only the random variable *X* [2] can be adopted.

To generate numbers according to the exponential distribution, the inverse distribution function method is applied.

If $F(x)$ is a cumulative distribution function of exponential distribution:

(4) $$F(x) = 1 - \exp^{-x}$$

Then:

(5) $$F^{-1}(r) = -\ln(1-r)$$

where: r – number generated with uniform distribution over the interval [0, 1].

The scheme for generating random numbers is as follows:

1. Generate number r_n with uniform distribution over the interval [0, 1],
2. Calculate the number:

(6) $$x_n = -\ln(1-r)$$

The presented method calculates a random variable with the parameter $c = 1$, however, if the parameter $c \neq 1$, the formula can be shown as:

(7) $$x_n = -\frac{1}{c}\ln(1-r) \quad c \neq 0$$

Weibull distribution generator

A random variable X has Weibull distribution with parameters (λ, α) if its probability distribution density is expressed by the formula [4]:

(8) $$f(t) = \begin{cases} 0 & \text{for} \quad t \le 0 \\ \lambda \cdot \alpha \cdot t^{\alpha-1} \cdot \exp[-\lambda \cdot t^{\alpha}] & \text{for} \quad t \le 0 \end{cases}$$

where $\lambda > 0$, $\alpha > 0$ are constants.

The cumulative distribution function of this distribution takes the form:

(9) $$F(x) = 1 - \exp[-\lambda \cdot x^{\alpha}], \quad x > 0$$

The inverse cumulative distribution function method leads to the following algorithm:

(1) Generate a random variable R with uniform distribution over the interval [0, 1],
(2) Calculate:

(10) $$X = \exp[\frac{1}{\alpha} \cdot \ln(R-1)]$$

For Weibull distribution, parameters b and v are most commonly applied and a conversion should also be used as follows:

(11) $$\alpha = \nu \quad \text{and} \quad \lambda = \frac{1}{b^{\nu}}$$

To generate the variables according to a normal distribution, a central limit theorem method should be applied. This method consists of generating a random variable X with a normal distribution and approximating it using aggregated independent random variables R_1, R_2, …, R_n with the same uniform distribution in the interval [0, 1]. Because the expected value $ER_i = ½$ and $D2R_i = 1/12$, thus for sufficiently large n random variable [5, 6, 10]:

(12) $$X = \frac{(R_1 + R_2 + ... + R_n) - \frac{n}{2}}{\sqrt{\frac{n}{12}}}$$

has a normal distribution N(0, 1). Assumed $n = 12$ (the so-called "right of the dozen"), then the denominator in formula 11 is equal to unity and there is no need to perform division.

Log-normal distribution generator

If a random variable X has a normal distribution with parameters (μ_x, δ_x), then the variable

Y= exp ^{x} has a log-normal distribution with parameters (μ_y, δ_y), the expected value of which and variance are equal, respectively, to: [6]

(13) $$\mu_y = \exp[\mu_x + \frac{\sigma_x^2}{2}]$$

(14) $$\sigma_y^2 = \exp[2\mu_x + \sigma_x^2] \cdot \exp[\sigma_x^2 - 1]$$

The following algorithm has been applied to generate the random variable Y for data μ_x and δ_x:

(1) Generate a random variable X according to a normal distribution with parameters N(0, 1),
(2) Calculate:

(15) $$Y = \exp[X \cdot \sigma_y + \mu_y]$$

Testing random number generators

Pseudorandom number generators are treated as being random generating devices such as a tossed coin when in use. The next number X_n produced by this generator is therefore taken as a random variable and the verification whether the generator produces random numbers with the desired probability distribution comes down to verifying whether the sequence $X_0, X_1, \ldots, X_{n-1}$ can be treated as a N – element simple sample of a specific population - so whether it is a series of independent random variables with the same probability distribution. [7]

Statistical tests designed to verify random number generators can be divided into two groups: independence tests and distribution compliance tests.

In the case of pseudorandom number generators as applied in electric power system reliability analyses, distribution compliance tests are crucial. Using Pearson's chi-squared tests and λ - Kolmogorov's tests, the hypothesis that a given sample of drawn numbers comes from a population with a given probability distribution has been verified. This verification has been made for the four pseudorandom number generators described earlier. Tests have been conducted to verify the given hypothesis for 50, 100 and 1000 drawn numbers.

Relations and symbols used for Pearson's test χ^2 (for all distributions) [8, 9]

(16) $$\chi^2 = \sum_{i=1}^{r} \frac{(n_i - n \cdot p_i)^2}{n \cdot p_i}$$

where: χ^2 – the statistical value determined, χ_{α}^2 – the statistical value read from Pearson's χ^2 distribution, n_i – number of elements in a given class interval, p_i – the probability that a random variable will take values of class 'i',

$$L_{ss} = r - k - 1 \tag{17}$$

where: L_{ss} – degrees of freedom, r – number of intervals, k – number of distribution parameters.

Relations and symbols used for the Kolmogorov's λ test:

$$n_{sk} = \sum_{i=1}^{n} n_i \tag{18}$$

where: n_{sk} – accumulated population size, n_i – number of elements in a given class interval,

$$F_n(x) = \frac{n_{sk}}{n} \tag{19}$$

where: $F_n(x)$ – empirical distribution function, n – sample size

$$F_1(x) = 1 - \exp[-\frac{t}{\theta}] \tag{20}$$

$$F_2(x) = 1 - \exp[-(\frac{t}{b})^{\nu}] \tag{21}$$

$$F_4(x) = 0,5 + \Phi(\frac{\lg(x) - m}{\sigma}) \tag{22}$$

where: $F_1(x)$ – a theoretical distribution function for exponential distribution, $F_2(x)$ – a theoretical distribution function for Weibull distribution, $F_3(x)$ – a theoretical distribution function for normal distribution (read from the tables), $F_4(x)$ – a theoretical distribution function for log-normal distribution, T – class interval end, θ – mean value.

$$D = \sup\left|F_n(x) - F(x)\right| \tag{23}$$

$$\lambda = D \cdot \sqrt{n} \tag{24}$$

where: λ - statistical value, λ_α - the statistical value for the statistical significance α = 0,05, λ_α = 1,358.

The article, due to its volume, includes test results (without detailed calculations) and a graphical presentation of empirical distributions for individual analyses.

Table 1. Results of calculations for the exponential distribution generator

	50 generated values	100 generated values	1000 generated values
Kolmogorov's λ test	λ = 0,35 $\lambda < \lambda_\alpha$ The following hypothesis should be adopted	λ = 0,5 $\lambda < \lambda_\alpha$ The following hypothesis should be adopted	λ = 0,158 $\lambda < \lambda_\alpha$ The following hypothesis should be adopted
Pearson's χ^2 test	χ^2 = 2,62 χ^2_α = 9,49 $\chi^2 < \chi^2_\alpha$ The following hypothesis should be adopted	χ^2 = 0,95 χ^2_α = 5,99 $\chi^2 < \chi^2_\alpha$ The following hypothesis should be adopted	χ^2 = 1,61 χ^2_α = 11,07 $\chi^2 < \chi^2_\alpha$ The following hypothesis should be adopted

Exponential distribution generator

On the basis of data obtained with the use of an exponential distribution generator, tests have been performed to verify the hypothesis that a given generator produces numbers according to exponential distribution. Table 1 summarises the results obtained from the tests.

For the analysis, a distribution with the parameter λ = 0,1542 has been adopted. Using the developed simulation software, 50, 100 and 1000 random number values have been generated.

On the basis of empirical data, a hypothesis on the exponential distribution of generated values has been established.

Below, Figures 1, 2 and 3 show the graphs for matching the theoretical distribution to empirical values, developed in Statistica software for 50, 100 and 1000 generated values, respectively.

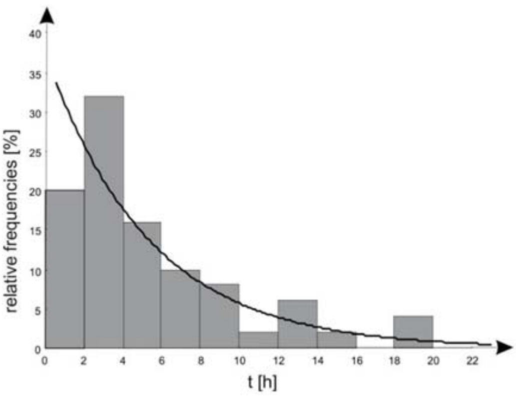

Fig. 1. Empirical distribution determined by simulation for 50 generated values according to exponential (bar chart) and theoretical (solid line) distribution

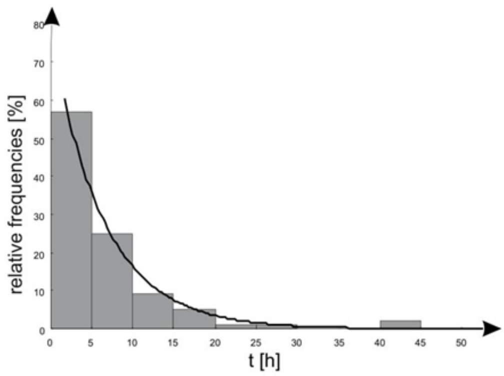

Fig. 2. Empirical distribution determined by simulation for 100 generated values according to exponential (bar chart) and theoretical (solid line) distribution

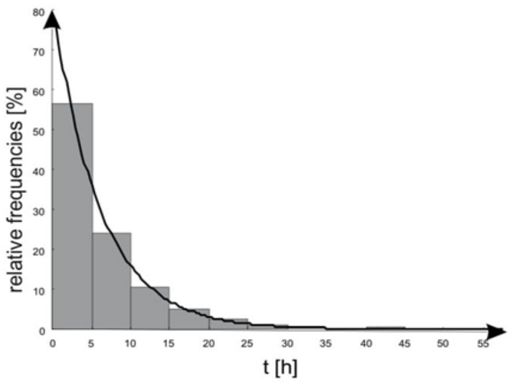

Fig. 3. Empirical distribution determined by simulation for 1000 generated values according to exponential (bar chart) and theoretical (solid line) distribution

Normal distribution generator

On the basis of data obtained with the use of a normal distribution generator, tests have been performed to verify the hypothesis that a given generator produces numbers according to normal distribution. Table 2 summarises the results obtained from the tests.[10]

The distribution with the parameters *m* = 17,8699 *δ* = 8,7684 has been adopted for analysis. Using the simulation software, 50, 100 and 1000 random number values have been generated.

On the basis of empirical data, a hypothesis on a normal distribution of generated values has been established.

Table 2. Results of calculations for the normal distribution generator

	50 generated values	100 generated values	1000 generated values
Kolmogorov's λ test	$\lambda = 0{,}71$ $\lambda < \lambda_\alpha$ hypothesis accepted	$\lambda = 0{,}4$ $\lambda < \lambda_\alpha$ hypothesis accepted	$\lambda = 0{,}949$ $\lambda < \lambda_\alpha$ hypothesis accepted
Pearson's χ^2 test	$\chi^2 = 9{,}09$ $\chi^2_\alpha = 7{,}81$ $\chi^2 > \chi^2_\alpha$ hypothesis rejected	$\chi^2 = 1{,}77$ $\chi^2_\alpha = 5{,}99$ $\chi^2 < \chi^2_\alpha$ hypothesis accepted	$\chi^2 = 76$ $\chi^2_\alpha = 5{,}99$ $\chi^2 > \chi^2_\alpha$ hypothesis rejected

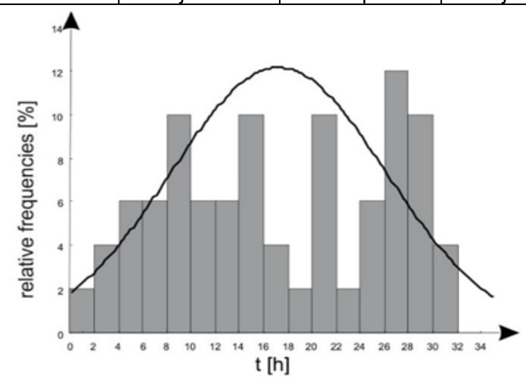

Fig. 4. Empirical distribution determined by simulation for 50 generated values according to normal (bar chart) and theoretical (solid line) distribution

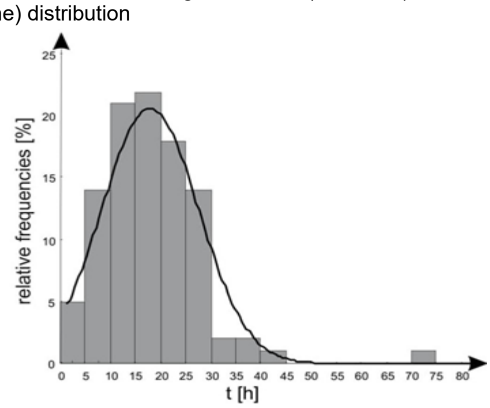

Fig. 5. Empirical distribution determined by simulation for 100 generated values according to normal (bar chart) and theoretical (solid line) distribution

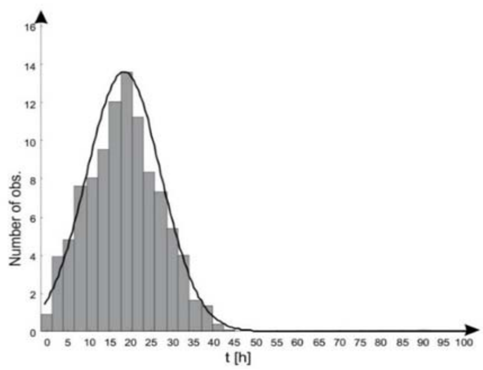

Fig. 6. Empirical distribution determined by simulation for 1000 generated values according to normal (bar chart) and theoretical (solid line) distribution

Figures 4, 5 and 6 show the graphs for matching the theoretical distribution to empirical values, developed in Statistica software for 50, 100 and 1000 generated values, respectively.

Weibull distribution generator

On the basis of data obtained with the use of a Weibull distribution generator, tests have been performed to verify the hypothesis that a given generator produces numbers according to Weibull distribution. Table 3 summarises the results obtained from the tests.

The distribution with the parameters *v* = 4,3, *b* = 12 has been adopted for analysis. Using the simulation software, 50, 100 and 1000 random number values have been generated. On the basis of empirical data, a hypothesis on the Weibull distribution of generated values has been established.

Table 3. Results of calculations for the Weibull distribution generator

	50 generated values	100 generated values	1000 generated values
Kolmogorov's λ test	$\lambda = 1{,}343$ $\lambda < \lambda_\alpha$ hypothesis accepted	$\lambda = 0{,}4$ $\lambda < \lambda_\alpha$ hypothesis accepted	$\lambda = 0{,}71$ $\lambda < \lambda_\alpha$ hypothesis accepted
Pearson's χ^2 test	$\chi^2 = 21{,}37$ $\chi^2_\alpha = 5{,}99$ $\chi^2 > \chi^2_\alpha$ hypothesis rejected	$\chi^2 = 5{,}45$ $\chi^2_\alpha = 18{,}31$ $\chi^2 < \chi^2_\alpha$ hypothesis accepted	$\chi^2 = 10{,}29$ $\chi^2_\alpha = 11{,}07$ $\chi^2 < \chi^2_\alpha$ hypothesis accepted

	50 generated values	100 generated values	1000 generated values
Kolmogorov's λ test	$\lambda = 1{,}343$ $\lambda < \lambda_\alpha$ hypothesis accepted	$\lambda = 0{,}4$ $\lambda < \lambda_\alpha$ hypothesis accepted	$\lambda = 0{,}71$ $\lambda < \lambda_\alpha$ hypothesis accepted
Pearson's χ^2 test	$\chi^2 = 21{,}37$ $\chi^2_\alpha = 5{,}99$ $\chi^2 > \chi^2_\alpha$ hypothesis rejected	$\chi^2 = 5{,}45$ $\chi^2_\alpha = 18{,}31$ $\chi^2 < \chi^2_\alpha$ hypothesis accepted	$\chi^2 = 10{,}29$ $\chi^2_\alpha = 11{,}07$ $\chi^2 < \chi^2_\alpha$ hypothesis accepted

Figures 7, 8 and 9 show the graphs for matching the theoretical distribution to empirical values, developed in Statistica software for 50, 100 and 1000 generated values, respectively.

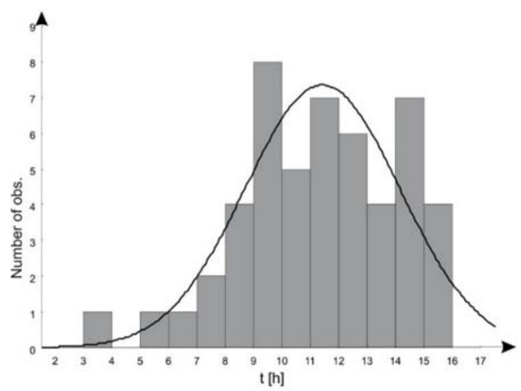

Fig. 7. Empirical distribution determined by simulation for 50 generated values according to Weibull (bar chart) and theoretical (solid line) distribution

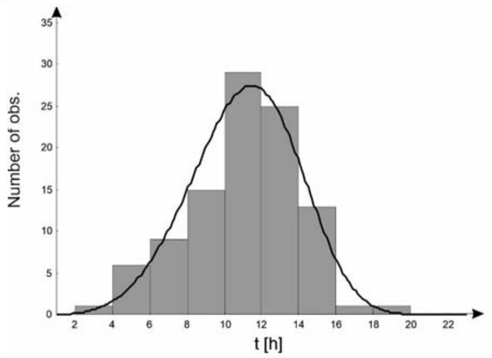

Fig. 8. Empirical distribution determined by simulation for 100 generated values according to Weibull (bar chart) and theoretical (solid line) distribution

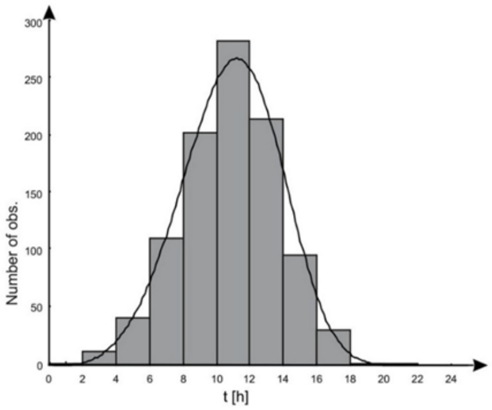

Fig. 9. Empirical distribution determined by simulation for 1000 generated values according to Weibull (bar chart) and theoretical (solid line) distribution

Log-normal distribution generator

On the basis of data obtained with the use of the log-normal distribution generator, tests have been performed to verify the hypothesis that a given generator produces numbers according to normal distribution. Table 4 summarises the results obtained from the tests.

The distribution with the parameters m = 1,99, δ = 0,05 has been adopted for analysis. Using the simulation software, 50, 100 and 1000 random number values have been generated.

On the basis of empirical data, a hypothesis on the log-normal distribution of generated values has been established.

Table 4. Results of calculations for the log-normal distribution generator

	50 generated values	100 generated values	1000 generated values
Kolmogorov's λ test	λ = 0,495 $\lambda < \lambda_\alpha$ hypothesis accepted	λ = 0,30 $\lambda < \lambda_\alpha$ hypothesis accepted	λ = 0,63 $\lambda < \lambda_\alpha$ hypothesis accepted
Pearson's χ^2 test	χ^2 = 13,96 χ^2_α = 19,68 $\chi^2 < \chi^2_\alpha$ hypothesis accepted	χ^2 = 6,28 χ^2_α = 12,59 $\chi^2 < \chi^2_\alpha$ hypothesis accepted	χ^2 = 8,65 χ^2_α = 14,07 $\chi^2 < \chi^2_\alpha$ hypothesis accepted

Figures 10, 11 and 12 show the graphs for matching the theoretical distribution to empirical values, developed in Statistica software for 50, 100 and 1000 generated values, respectively.

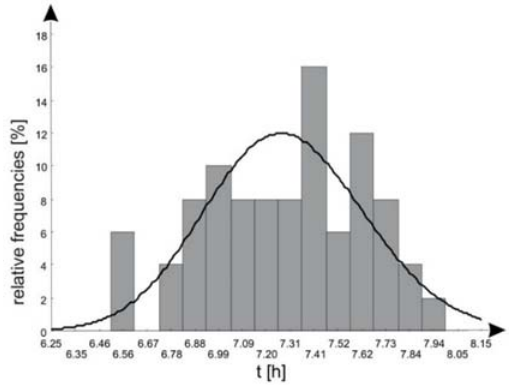

Fig. 10. Empirical distribution determined by simulation for 50 generated values according to log-normal (bar chart) and theoretical (solid line) distribution

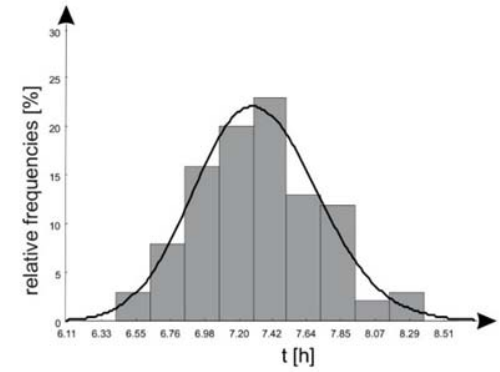

Fig. 11. Empirical distribution determined by simulation for 100 generated values according to log-normal (bar chart) and theoretical (solid line) distribution

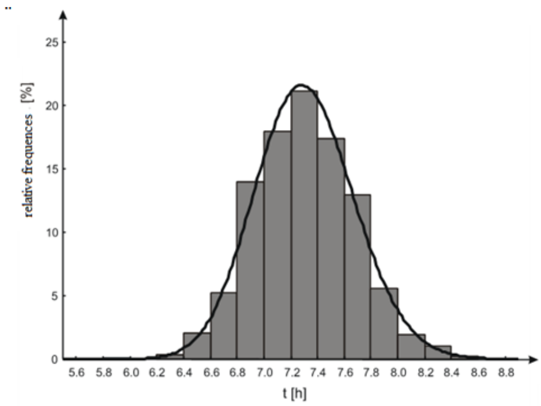

Fig. 12. Empirical distribution determined by simulation for 1000 generated values according to log-normal (bar chart) and theoretical (solid line) distribution

Conclusion

By way of application of the mathematical relations described in the article, it is possible to generate random numbers according to a given distribution with specific parameters. With any software modelling the operation of electric power system devices, it is possible to program their operation according to the assumed distribution. This provides an opportunity to perform wider and more effective reliability analyses.

Only the results for the generated 50, 100 and 1000 values are included in the article, due to its volume. An analysis has also been performed for 500 and 10,000 results. As shown, matching the generated empirical distributions to the theoretical value of the probability density function is better for a large random sample.

For over 100 generated values, according to the given distribution, the results are considered satisfactory for each of these. Indeed, during a computer simulation performed for electric power systems, we obtain a number of results between 100 and 1000 cases and these values can be considered as correctly generated.

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