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An optimization-based method for tuning PI controllers for systems subject to control and output constraints

Abstract. *This article presents an optimization-based method to tuning Proportional-Integral (PI) controllers for systems subject to actuator saturation and constraints on the controlled variable. The theory of invariant sets is used to treat constraints and a polytopic model is adopted to account for actuator saturation. For the numerical validation of the method, four examples are presented. The results show that the method can establish a tuning that does not violate any constraint.*

Streszczenie. *W artykule przedstawiono opartą na optymalizacji metodę strojenia regulatorów proporcjonalno-całkujących (PI) dla układów podlegających nasyceniu elementów wykonawczych i ograniczeniom na regulowaną zmienną. Teoria zbiorów niezmienniczych jest wykorzystywana do leczenia ograniczeń, a model politopowy jest przyjmowany w celu uwzględnienia nasycenia elementu wykonawczego. W celu numerycznej walidacji metody przedstawiono cztery przykłady. Wyniki pokazują, że metoda może ustanowić strojenie, które nie narusza żadnego ograniczenia.* ***(Oparta na optymalizacji metoda strojenia regulatorów PI dla systemów podlegających ograniczeniom sterowania i wydajności)****.*

Keywords: PI control, constraints, saturation, invariant sets
Słowa kluczowe: Sterowanie PI, nasycenie, ograniczenia sterowania

Introduction

The proportional-integral-derivative controller (PID) is an algorithm widely used in the industry to control variables such as speed, pressure, and level. Some studies estimated that 95 % of the control loops use PID controllers, most of them in the proportional-integral (PI) form [1].

One of the difficulties in using PID controllers is to find an appropriate tuning. Usually, the Ziegler-Nichols rules are used [23], which are based on a simple model whose parameters, in many cases, can be obtained through simple experiments on the plant [2].

Because Ziegler-Nichols rules result quite often in undesirable oscillatory response and poor robustness, other rules, also based on typical dynamic models have been developed such as Cohen and Coon, integral absolute error, and integral time-weighted absolute error, etc. There are also tuning methods based on bio-inspired optimization. In the article [16], a new PI controller tuning method is proposed based on modifying the particle swarm optimization. Its results are reasonable compared to the classical methods presented in the article. In the article [4], a PI controller based on the ant colony optimization is tuned to a shunt active power filter; the controller found has demonstrated good performance for harmonic elimination and reactive power compensation.

However none of these rules explicitly deal with important aspects present in any industrial processes: constraints on controlled variables and actuator saturation [20].

Actuator saturation, if not properly dealt with, can result in poor controller performance or even closed-loop instability. To deal with this problem, anti-windup compensation was initially used, which modifies the controller that will act when saturation occurs, aiming to recover, as quickly as possible, the performance designed for the system without saturation. More sophisticated anti-windup techniques later emerged, which can guarantee stability and performance under saturation [19]. Note that the first anti-windup methods, anti-windup compensator, are simple to implement, but do not guarantee stability, whereas modern anti-windup has a complex implementation, but it does guarantee stability.

Some PID tuning methods explicitly deal with constraints. In [20] an analytical design of PI controllers is proposed for optimal regulatory control of open-loop unstable processes under operational constraints. It deals with the three common operational constraints related to the process variable, manipulated variable and its rate of change. To derive analytical design relations, the constrained optimal control problem in time domain was transformed into an unconstrained optimization problem in a new parameter space.

Another way to solve control problems under constraints is to use polyhedral Lyapunov functions and positively invariant sets. This combination guarantees stability and local performance in closed-loop, respecting constraints even in the presence of amplitude-limited disturbances and noise.

These sets have been intensively used to solve problems of regulation of linear systems under constraints [21, 11, 6, 5]. In [22], invariant sets are used to address the problem of physical and safety constraints of multi-mass electrical drives, in combination with the classical LQR controller. The practical implementation of a piecewise affine PI controller using invariant sets was reported in [15] for first order systems. The maximal controlled invariant set, defined on the space formed by the tracking error and its integral is split into polyhedral regions, each one with associated PI gains. A tuning method using invariant sets was proposed in [17], which addresses constant reference tracking for SISO systems subject to state and control constraints via a PI control law with a feedforward term. The invariant polyhedral sets and controller gains that guarantee compliance with constraints are computed via the bilinear programming approach proposed in [7].

In this work we propose a tuning method for PI controllers for discrete-time systems of any order subject to constraints on the controlled variable and actuator saturation. Conditions are presented for a polyhedron defined on an extended state space to be positively invariant with respect to the closed-loop system with the PI controller. As in [7], these conditions are translated into an optimization problem with bilinear constraints, whose solution provides the controller gains which minimize a cost function that accounts for the speed of the tracking response and disturbance rejection. The performance of the controller is assessed, and an analysis of the main features of the method is performed through numerical experiments.

The paper is organized as follows: in section Invariant sets and control under constraints, we present the main concepts and results on polyhedral invariant sets, which will be used to enforce output constraints under actuator saturation. In particular, we present sufficient conditions under which a given polyhedral set is positively invariant with respect to a closed-loop system under output feedback, actuator saturation, and constraints on the controlled output. Such results are used in section Design of PI controller with output constraints and input saturation to propose a PI controller scheme and an optimization problem whose solution delivers controller gains that enforce the constraints and a polyhedral invariant set which is an approximation of the set of admissible initial states. Numerical experiments are presented in section Numerical Examples and conclusions are drawn in section Conclusion.

Invariant sets and control under constraints

Consider the following linear, time-invariant,Single-Input and Multiple-Output (SIMO) discrete-time system:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] + Ed[k], \\ y[k] &= Cx[k], \end{aligned} \tag{1}$$

where $k \in \mathbb{N}$ is the time index, $x[k] \in \mathbb{R}^n$ is the state, $u[k] \in \mathbb{R}$ is the control input, $d[k] \in \mathbb{R}$ is the disturbance, and $y[k] \in \mathbb{R}^p$ is the measured output.

The disturbance $d[k]$ is unknown but bounded to a compact polyhedron containing the origin:

$$d[k] \in \Phi = \{d : |Wd[k]| \leq \mathbf{1}\}, \tag{2}$$

where $W \in \mathbb{R}$, inequalities are element-wise, and $\mathbf{1}$ is a vector of suitable dimensions whose elements are all equal to 1.

The system is subject to actuator saturation:

$$u[k] \in \Omega_u = \{-u_{max} \leq u[k] \leq u_{max}\}. \tag{3}$$

Considering static output feedback:

$$u[k] = sat(Ky[k]) = sat(KCx[k]), \tag{4}$$

where $sat(.)$ is the saturation function, defined by:

$$sat(f) = \begin{cases} f, & \text{if} \quad -u_{max} \leq f \leq u_{max}, \\ u_{max}, & \text{if} \quad f > u_{max}, \\ -u_{max}, & \text{if} \quad f < -u_{max}. \end{cases} \tag{5}$$

the closed-loop system takes the form:

$$x[k+1] = Ax[k] + Bsat(KCx[k]) + Ed[k]. \tag{6}$$

System (1) is subject to constraints on the output:

$$y[k] \in \Omega_y = \{|Qy[k]| \leq \mathbf{1}\}, \tag{7}$$

which, from the output equation in (1), become constraints on the state:

$$x[k] \in \Omega_x = \{|QCx[k]| \leq \mathbf{1}\}. \tag{8}$$

For simplicity, we will deal with sets that are symmetrical with respect to the origin. The results presented can, however, be easily extended to non-symmetrical polyhedra.

State and output constraints can be satisfied if the initial state is contained in a positively invariant set, defined below.

Definition 1 *A set $\Omega \subset \mathbb{R}^n$ is said to be* positively invariant *with respect to system* (6) *if $\forall x[k] \in \Omega, x[k+1] \in \Omega$, $\forall k \geq 0$, $\forall d[k] \in \Phi$.*

Linear constraints on state variables result in polyhedral sets in the state space, such as:

$$\Omega = \{x : |Lx[k]| \leq \mathbf{1}\},$$

where $L \in \mathbb{R}^{l \times n}$.

As it will become clear later, the problem of tuning a PI-controller for linear systems subject to output constraints and actuator saturation can be recast as that of computing the controller gains and an associated polyhedral set $\Omega \subset \Omega_x$ which is positively invariant with respect to the closed-loop system (6).

Due to the saturation function, system (6) is nonlinear, making it difficult to directly treat invariance. For this reason, many tractable approximated models have been proposed to this end. Here we will use the polytopic model described in [18], which is based on the following result:

Lemma 1 *[18] Consider $f \in \mathbb{R}$ and $h \in \mathbb{R}$. If $-u_{max} \leq h \leq u_{max}$ ($|h| \leq u_{max}$), then it follows that:*

$$sat(f) \in Co\{f, h\} = \{\lambda_1 f + \lambda_2 h,\ 0 \leq \lambda_1, \lambda_2 \leq 1,\ \lambda_1 + \lambda_2 = 1\},$$

where Co stands for the *convex hull*.

Consider now the following polyhedron:

$$S(H_{sat}, u_{max}) = \{x \in \mathbb{R}^n; |H_{sat}x[k]| \leq u_{max}\},$$

Then, from Lemma 1, since $f = Ky[k] = KCx[k]$, then if $x[k] \in S(H_{sat}, u_{max})$ it follows that:

$$u[k] = sat(KCx[k]) \in Co\{KCx[k], H_{sat}x[k]\},$$

where $H_{sat} \in \mathbb{R}^{1 \times n}$ is an auxiliary matrix to compose the saturation model.

As a consequence, system (6) can be approximated by the following polytopic model:

$$x[k+1] = \lambda_1 A_1 x[k] + \lambda_2 A_2 x[k] + Ed[k], \tag{9}$$

with $\lambda_1 + \lambda_2 = 1$, $0 \leq \lambda_1, \lambda_2 \leq 1$, and

$$A_1 = A + BKC, \quad A_2 = A + BH_{sat}. \tag{10}$$

From this formulation, we can establish conditions for positive invariance of a polyhedron $\Omega = \{x : Lx \leq \mathbf{1}\}$ with respect to the polytopic model (9).

Proposition 1 *The polyhedron $\Omega = \{x : Lx \leq \mathbf{1}\}$ is positively invariant with respect to the system* (9) *if there exist matrices Z, H_k, H_h, and H_{sat} such that:*

$$\begin{aligned} H_k L &= L(A + BKC), \\ ZW &= LE, \\ \|[H_k \ \ Z]\|_\infty &\leq 1, \\ H_h L &= L(A + BH_{sat}), \\ \|[H_h \ \ Z]\|_\infty &\leq 1. \end{aligned} \tag{11}$$

where $\|P\|_\infty$ stands for the induced infinity norm of matrix $P \in \mathbb{R}^{n \times m}$, with elements p_{ij}, given by $\|P\|_\infty = max_i \sum_{j=1}^{m} |p_{ij}|$.

Proof: Consider the model (9). Then, from condition (11):

$$\begin{aligned} |Lx[k+1]| &= |\lambda_1 LA_1 x[k] + \lambda_2 LA_2 x[k] + LEd[k]| \\ &= |\lambda_1 H_k Lx[k] + \lambda_2 H_h Lx[k] + ZWd[k]| \end{aligned}$$

Considering $|Lx[k]| \leq \mathbf{1}$ and $|Wd[k]| \leq \mathbf{1}$:

$$\begin{aligned} |Lx[k+1]| &\leq |\lambda_1 H_k|\mathbf{1} + |\lambda_2 H_h|\mathbf{1} + (\lambda_1 + \lambda_2)|Z|\mathbf{1} \\ &= \lambda_1 |H_k \mathbf{1} + Z\mathbf{1}| + \lambda_2 |H_h \mathbf{1} + Z\mathbf{1}|. \end{aligned}$$

Since $P\mathbf{1} \leq \mathbf{1}$ is equivalent to $\|P\|_\infty \leq 1$, we finally have that: $|Lx[k+1]| \leq \lambda_1 \|[H_k \ \ Z]\|_\infty + \lambda_2 \|[H_k \ \ Z]\|_\infty \leq 1$. This proves that $x[k+1] \in \Omega$ if $x[k] \in \Omega$, then, the positive invariance of Ω w.r.t. (9). □

According to Lemma 1, the polytopic model is valid only if $x[k] \in S(H_{sat}, u_{max})$. This condition can be imposed if the positive invariant set Ω is included in $S(H_{sat}, u_{max})$. The following proposition establishes conditions under which Ω is positively invariant with respect to the nonlinear system (6).

Proposition 2 *The polyhedron* $\Omega = \{x \ : \ Lx \leq \mathbf{1}\}$ *is positively invariant with respect to the system* (6) *if there exist matrices* Z*,* H_k*,* H_h*, and* H_{sat} *satisfying* (11) *and a matrix* J *such that:*

$$\begin{cases} JL = H_{sat}, \\ \|J\|_\infty \leq u_{max}. \end{cases} \tag{12}$$

proof 1 *According to the so-called* extended *Farkas' lemma (see e.g. [8]), condition* (12) *implies* $\Omega \subset S(H_{sat}, u_{max})$*, which guarantees that* $x[k] \in S(H_{sat}, u_{max})$ *with respect to the polytopic model* (9) *if* Ω *is positively invariant. But this is guaranteed by conditions* (11)*. Finally, in [18] it is proven that, in this case, if* Ω *is positively invariant with respect to* (9)*, then, it is also positively invariant with respect to* (6)*.*

Clearly, output constraints (7) can be satisfied if the positively invariant set Ω is included in the set Ω_x (8). This condition can also be enforced by using the Extended Farkas' Lemma, as follows:

Proposition 3 $\Omega \subset \Omega_x$ *if, and only if, there exists a matrix* $H_x \in \mathbb{R}^{n \times l}$*, such that:*

$$\begin{cases} H_x L = QC, \\ \|H_x\|_\infty \leq 1. \end{cases} \tag{13}$$

Design of PI controller with output constraints and input saturation

In the previous section we presented conditions under which a polyhedral set is positively invariant with respect to a linear system under saturated static output feedback. In this section we propose an augmented state-space formulation that allows to compute the gains and an associated positively invariant polyhedral set (which stands for a guaranteed set of admissible initial states) of a PI controller which enforces output constraints under actuator saturation.

PI Controller in an Augmented State Space

Consider a discrete-time PI controller as shown in Fig. 1, represented by:

$$f[k] = K_p e[k] + K_i v[k], v[k+1] = v[k] + T_s e[k], \tag{14}$$

where T_s is the sampling period, and $e[k] = w[k] - y[k]$.

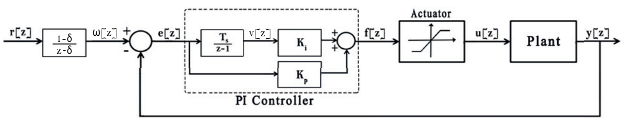

Fig. 1. Closed-loop system with PI controller and actuator saturation.

We assume that the reference signal has a first-order dynamics as follows:

$$\omega[k+1] = \delta\omega[k] + (1-\delta)r[k], \tag{15}$$

with $0 \leq \delta \leq 1$.

As it will become clear in the sequel, this strategy is used to soften the reference signal, as it can undergo abrupt amplitude changes that, in order not to violate constraints, result in a conservative tuning of the PI controller. With the filter attenuating the variation of the reference, it is possible to obtain a less conservative tuning which enforces the constraints. One can see from (15) that $\omega[k]$ tends to $r[k] = r$ (constant) in steady-state.

System (1) considering $y[k]$ as a scalar, without the disturbance $d[k]$, and the PI controller (14) with the reference (15) can be represented in an augmented state space in the form:

$$\underbrace{\begin{bmatrix} x[k+1] \\ v[k+1] \\ \omega[k+1] \end{bmatrix}}_{\bar{x}[k+1]} = \underbrace{\begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ -CT_s & 1 & T_s \\ \mathbf{0} & 0 & \delta \end{bmatrix}}_{\bar{A}} \begin{bmatrix} x[k] \\ v[k] \\ \omega[k] \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{\bar{B}} u[k] + \underbrace{\begin{bmatrix} \mathbf{0} \\ 0 \\ (1-\delta) \end{bmatrix}}_{\bar{E}} r[k], \tag{16}$$

$$\underbrace{\begin{bmatrix} e[k] \\ v[k] \end{bmatrix}}_{\bar{y}[k]} = \underbrace{\begin{bmatrix} -C & 0 & 1 \\ \mathbf{0} & 1 & 0 \end{bmatrix}}_{\bar{C}} \begin{bmatrix} x[k] \\ v[k] \\ \omega[k] \end{bmatrix}, \tag{17}$$

$$\begin{aligned} \bar{x}[k+1] &= \bar{A}\bar{x}[k] + \bar{B}u[k] + \bar{E}r[k], \\ \bar{y}[k] &= \bar{C}\bar{x}[k]. \end{aligned} \tag{18}$$

The control law is given by:

$$u[k] = sat(f[k]) = sat\left(\underbrace{\begin{bmatrix} K_p & K_i \end{bmatrix}}_{\bar{K}} \begin{bmatrix} e[k] \\ v[k] \end{bmatrix} \right). \tag{19}$$

In view of (9), the polytopic model becomes:

$$\bar{x}[k+1] = \lambda_1(\bar{A}+\bar{B}\bar{K}\bar{C})\bar{x}[k] + \lambda_2(\bar{A}+\bar{B}H_{sat})\bar{x}[k] + \bar{E}r[k].$$

It one can notice that under this formulation, compared to the model (1), the reference signal $r[k]$ assumes artificially the role of a disturbance.

The system is also subject to constraints on the controlled output $y(k) = Cx(k)$ which translate into constraints on the state $x[k]$. In order to cope with the augmented model, we add constraints on $v[k]$ and $\omega[k]$, so that the augmented state vector $\bar{x}[k]$ must belong to a compact polyhedron:

$$\Omega_s = \{|\bar{Q}\bar{x}| \leq \mathbf{1}\}. \tag{20}$$

The definition of constraints on $v[k]$ and $\omega[k]$ will be discussed in the Numerical Examples section.

From the previous formulation, we can use the conditions derived in section Invariant sets and control under constraints, to find a solution to the following design problem:

Problem 1 *Given a system represented by* (1) *with* $d[k] = 0$*, compute the PI tuning (*K_p *and* K_i*) and an associated positively invariant polyhedron* Ω *with respect to the augmented system (18) such that if* $\bar{x}[0] \in \Omega$ *the trajectory of* $\bar{x}[k]$ *remains in* Ω *and satisfies the output constraints* (20)*.*

One can see that the augmented system (18) has the same structure as system (1). Then, based on the development of section Invariant sets and control under constraints, we are in position to establish conditions under which a solution to Problem 1 can be obtained.

Proposition 4 *Problem 1 has a solution if there exist matrices* $\bar{L}$*,* $\bar{K}$*,* $\bar{H}_k$*,* $\bar{H}_h$*,* $\bar{Z}$*,* $\bar{H_{sat}}$*,* $\bar{J}$*, and* $\bar{H}_s$*, of appropriate* *dimensions, such that:*

$$(21)\quad \begin{array}{c} \bar{H}_k\bar{L} = \bar{L}(\bar{A} + \bar{B}\bar{K}\bar{C}), \\ \bar{Z}\bar{W} = \bar{L}\bar{E}, \\ \left\|[\bar{H}_k \;\; \bar{Z}\,]\right\|_\infty \leq 1, \\ \bar{H}_h\bar{L} = \bar{L}(\bar{A} + \bar{B}\bar{H_{sat}}), \\ \left\|[\bar{H}_h \;\; \bar{Z}]\right\|_\infty \leq 1. \end{array}$$

$$(22)\quad \begin{cases} \bar{J}\bar{L} = \bar{H_{sat}}, \\ \left\|\bar{J}\right\|_\infty \leq u_{max}. \end{cases}$$

$$(23)\quad \begin{cases} \bar{H}_s\bar{L} = \bar{Q}, \\ \left\|\bar{H}_s\right\|_\infty \leq 1. \end{cases}$$

Remarks:

1. If there exists a matrix $\bar{L}$ satisfying conditions (21), (22), then, the polyhedral set $\bar{\Omega} = \{\bar{x} : \bar{L}\bar{x} \leq \mathbf{1}\}$ is positively invariant with respect to (18)-(19), hence, the trajectory of $\bar{x}[k]$ is assured to remain in $\bar{\Omega}$ if $x[0] \in \bar{\Omega}$.
2. Conditions (23) guarantee that $\bar{\Omega} \subset \Omega_s$, hence, that the output constraints (20) are satisfied. Since Ω_s is a bounded polyhedron, then $\bar{\Omega}$ is also bounded. Moreover, it is closed by definition, which implies that it is compact. In this case, as shown in [18], the closed-loop system is locally stable and $\bar{\Omega}$ is an estimate of the region of attraction. If, in addition, $\bar{\Omega}$ is contractive ($\bar{x}[k+1] \in \lambda\bar{\Omega}$ if $\bar{x}[k] \in \Omega$, with $0 \leq \lambda < 1$), then the closed-loop system is locally asymptotically stable.

BILINEAR OPTIMIZATION DESIGN STRATEGY

The algebraic conditions in Proposition 4 mostly carry products among pairs of matrix variables, including the control gains $\bar{K}$ to be synthesized, as well as other bilinear terms arising from the products between matrices and scalars or vectors.

Conditions (21), (22), and (23) can be appropriately considered as design constraints, and adapted nonlinear optimization techniques can be used to compute suitable control gains.

The proposed design strategy considers minimizing the reference filter parameter δ and maximizing K_i. Parameter δ directly relates to the speed of the system's response. As the filter (15) smooths the original reference, the smaller the value of δ, the faster the filtered reference tends to the desired reference, generally resulting in faster tracking response. As a result, one can expect a shorter settling time for constant reference signals. Maximizing the gain of the integral action K_i tends to decrease the integral of the error in disturbance rejection problems [2]. This way, we can choose the objective function so as to have a fast response for both the regulation and the reference tracking problem.

The following programming problem with bilinear constraints is proposed to find a solution to the PI control design under constraints:

$$(24)\quad \begin{array}{cc} \underset{(\bar{H}_k,\bar{H}_h,\bar{H}_s,\bar{H_{sat}},\bar{J},\bar{Z},\bar{L},\bar{K})}{min} & (w_1\delta + w_2\frac{1}{K_i}) \\ subject \quad to & (21),(22),(23) \\ f_l(.) \leq \varphi_l, l = 1,\ldots,\bar{l}, & \end{array}$$

The cost function weights the value of the filter parameter δ and the integral gain K_i. In general, a smaller value of δ results in a faster step response, whereas a larger value of K_i results in a faster disturbance rejection. Then, by choosing the weighting factors w_1 and w_2 the designer will be giving more importance to reference tracking ($w_1 > w_2$) or disturbance rejection ($w_1 < w_2$).

The additional constraints represented by $f_l(.) \leq \varphi_l$ correspond to lower and upper bounds imposed on the variables of the optimization problem $(\bar{H}_k, \bar{H}_h, \bar{H}_s, \bar{H_{sat}}, \bar{J}, \bar{Z}, \bar{L}, \bar{K})$. $\bar{l}$ is a non-negative integer which depends on the number of variables. These bounds are intended to reduce the search space of the optimization algorithm, making it easier to reach a solution.

The constraints in (24) involve products between matrix variables, and then we have bilinear constraints. The problem is not bilinear though, because the cost function is not bilinear. Bilinear constraints are not convex, then, the optimization problem can be hard to solve because many local minima can exist. The KNITRO solver, used in this work, proved to be efficient and robust for dealing with bilinear constraints. Even though it does not guarantee to find globally optimal solutions, local minima are found upon convergence. For a more in-depth discussion on the solution of the optimization problem, we refer the reader to [7].

One advantage of this approach compared to other approaches which use invariant sets, e.g. [15] is that the complexity of the polyhedron, represented by the number of rows of matrix L, can be fixed *a priori*. Thus, the number of rows of L is the only tunable parameter. A necessary condition for boundedness of the polyhedron is that it be greater than or equal to the number of states.

The use of bilinear programming to compute invariant sets and associated controllers for systems subject to constraints was first proposed in [7], where static state and output feedback controllers are designed to solve a regulation problem under state and control constraints, but with saturation avoidance. Polytopic model of saturation has been used in [14] to analyse positive invariance of polyhedral sets for discrete-time systems in the context of the so-called δ-operator.

Numerical Examples

KNITRO solver from the NEOS Server [10], a free internet-based service for solving numerical optimization problems, was used to generate the results we report now. The default solver configurations were used, together with the multi-algorithm option. Also, the following upper and lower bounds (-10^3,10^3) for all the elements of the matrices of the bilinear programming problem.

Four tests were carried out to explore the variety of potential applications of the proposed tuning method. A comparison was made between the proposed method and other PI controller tuning methods in the first test. In the second test, we explore the effect on the time response of the choice of the weighting factors w_1, w_2 in the optimization criterion. In the third test, we explore the application of the proposed method to systems with small and long dead-time. A comparison is made between the proposed method and other PI controller tuning methods applied to an open-loop unstable plant in the fourth test. For all tests, the weighting factors were adjusted to obtain a fast response leading to saturation of the control signal in order to explore the potential of the proposed method.

Note that there is still no systematic way to determine a bound to $v[k]$. A lower bound equal to the bound on the output can be imposed to ω[k].

Example 1: Comparison with other techniques

In this example we compare the proposed method to the PI controller tuning method for constrained systems proposed in [15]. There, a piecewise affine proportional-integral (PWA-PI) controller algorithm is designed based on invariant sets and multiparametric programming for constrained systems. The method has been implemented in a programmable logic controller to control an industrial level plant and analyze its behaviour. To show the performance of the designed controller, performance indexes (such as mean square error, Goodhart, overshoot, and settling time) were used, which show that the proposed method resulted in a better response than Ziegler–Nichols (Z–N) rules. Despite the promising results, PWA-PI application is limited to first-order systems. In contrast, our proposed method is applicable to systems of any order. Moreover, the design and implementation of the PWA-PI controller are more complex than those of a standard PI such as in the proposed method.

Consider the following linear continuous-time system, borrowed from [15]. This model was obtained by system identification around $y = 50$. Therefore, the variables of the model are the deviations around this operating point. Follow the transfer function:

$$G(s) = \frac{0.285}{s + 0.2903} \tag{25}$$

A discrete-time model was obtained using the zero-order-holder discretization method with sample period $T_s = 0.5\ s$. A realization in the state space of the discrete-time model is as follows:

$$\begin{aligned} x[k+1] &= 0.8649x[k] + 0.1326u[k], \\ y[k] &= x[k]. \end{aligned} \tag{26}$$

The system is subject to the following constraints:

$$\begin{aligned} -50 \leq y[k] \leq 50 \\ -50 \leq u[k] \leq 50 \end{aligned} \tag{27}$$

and the additional constraints to cope with the augmented system (20):

$$\begin{aligned} -300 \leq v[k] \leq 300 \\ -50 \leq \omega[k] \leq 50 \\ -50 \leq e[k] \leq 50 \end{aligned} \tag{28}$$

The additional constraints on $e[k]$ were included for the sake of comparison with the technique in [15]. By applying the bilinear design strategy (24), the results below were obtained. To verify the efficiency of the technique, we compared the closed-loop response with that obtained by the PWA-PI approach.

The number of rows of $\bar{L}$ was fixed to 20.We aim here at a fast tracking response, that can be obtained with a small value of δ. Hence, we choose $w_1 = 15$, $w_2 = 1$, giving more weight to δ in the optimization criterion. The controller parameters found were K_p = 6.5131, K_i = 1.2206, and δ = 0.7795.

It is possible to see through Fig. 2 that the constraints have not been violated.

Fig. 3 shows the control input and the plant output for a step change in the reference. One can notice that the control action saturates. The setpoint was changed from 0 to 40 in the linear model, equivalent to 50 to 90 if one considers a start from the operating point, in 5 seconds time, being the setpoint close to the output constraint. Table 1 shows

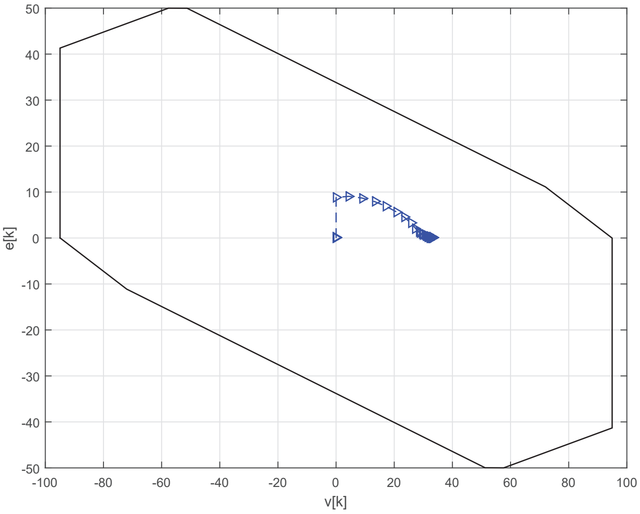

Fig. 2. Example 1: projection of the invariant polyhedron onto $v \times e$ and the reference tracking trajectory.

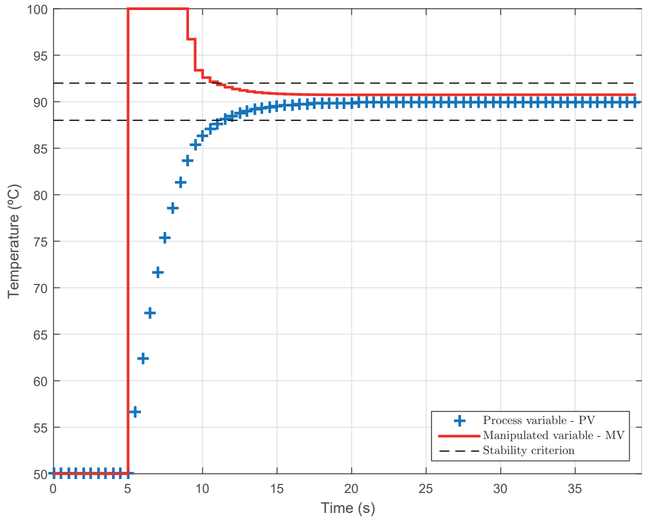

Fig. 3. Example 1: system response.

a comparison of performance indexes considering 5 criteria: overshoot, settling time $t_{s5\%}$, saturation interval, Integral of the Absolute Error (IAE), and Integral of the Absolute Error multiplied by Time (ITAE).

The proposed PI controller resulted in a response without overshoot, differently from the PWA-PI controller that resulted in a 5.5 % overshoot. Considering the 5 % settling time criterion ($t_{s5\%}$), the output obtained with the proposed controller settled in 6.5 s, whereas that obtained with the PWA-PI controller settled in 10 s. Saturation occurred when there was a change in the setpoint. In this situation, the output of the proposed PI controller saturated for 4 s, whereas that of the PWA-PI controller saturated for 8 s. Concerning the IAE an improvement of 68 % was achieved by the proposed controller, compared to the PWA-PI. A significant reduction of ITAE can also be observed. Thus, for the criteria eval-

Controller	Proposed PI	PWA-PI
Overshoot (%)	0	5.5
$t_{s5\%}$ (s)	6.5	10
Saturation interval (s)	4	8
IAE	3.54	5.20
ITAE	11.19	25.18

Table 1. Comparison of the performance evaluation of the plant controlled by proposed PI and PWA-PI.

uated, the proposed PI controller delivered a better performance than the PWA-PI controller, without violating any constraint.

Example 2: Weighting of the objective function

In this test we assess the effect in the time response of the choice of the weighting factors w_1, w_2 in the optimization criterion. We consider system (26), output and input constraints (27), and the additional constraints for the augmented system:

$$\begin{array}{c} -300 \leq v[k] \leq 300 \\ -50 \leq \omega[k] \leq 50 \end{array} \tag{29}$$

We compare two different tunings:

1. **T1:** w_1=10, w_2=1, aiming at a small value of δ.
2. **T2:** w_1=1, w_2=10, aiming at a large value of K_i.

In both tests, the number of rows of $\bar{L}$ was fixed to 20. The following results were obtained, respectively:

1. **T1:** K_p = 8.976, K_i = 3.5961, $\delta = 0.8622$.
2. **T2:** K_p = 9.4887, K_i = 6.7978, $\delta = 0.8999$.

Leaving from $y = 0$ a unit step reference is applied, and at $k = 80$ a disturbance of amplitude 1 is applied to the input.

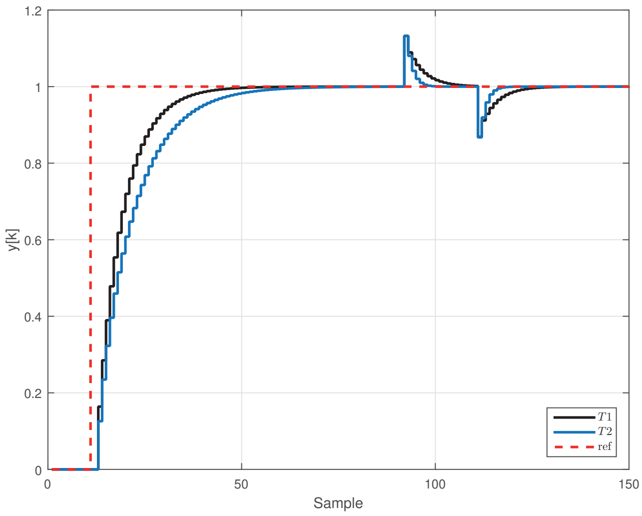

Fig. 4. Example 2: system response with disturbance.

From Figure (4) one can see that, as expected, tuning T1 resulted in a faster reference tracking response (larger δ), whereas tuning T2 resulted in a better disturbance rejection (larger K_i). In this way, by properly choosing the weighting factors w_1, w_2 the designer can decide which controller performance is more significant; regulation or reference tracking.

Example 3: First Order Plus Dead Time (FOPDT) system

The main objective of this example is to show that the proposed method has the potential to find a PI tuning for dead-time systems without the need for additional modifications. To this end, two examples will be presented, one with a small dead-time and other with a large dead-time. The characteristics and difficulties encountered for each test are different, as it will be seen in the sequel.

Small dead-time

Consider the FOPDT system, borrowed from [13], with transfer function:

$$G(s) = \frac{e^{-s}}{8s+1} \tag{30}$$

The sampling period was chosen as $T_s = 0.4\ s$. For the discretization, the zero-order holder method was used. The discrete-time transfer function is given by:

$$G(z) = \frac{0.02469z + 0.02408}{z^3(z-0.9512)}, \tag{31}$$

and a state space realization in the observable canonical form was computed to the fourth order system. The system is subject to the following constraints:

$$\begin{array}{c} -2 \leq y[k] \leq 2 \\ -1.2 \leq u[k] \leq 1.2 \end{array} \tag{32}$$

and the additional constraints:

$$\begin{array}{c} -2 \leq x_i[k] \leq 2,\ for\ i = 1, \cdots, 4; \\ -10 \leq v[k] \leq 10 \\ -10 \leq \omega[k] \leq 10 \end{array} \tag{33}$$

By applying the bilinear design strategy (24), the following results were obtained. The number of rows of $\bar{L}$ was fixed to 18, and the weighting factors were chosen to be w_1 = 3, w_2=13. The obtained controller gains were K_p = 1.1760, K_i = 0.1463, and $\delta = 4.266 \times 10^{-9}$, meaning that the reference signal has barely been filtered. Given that a small value of δ is already obtained, a value of w_2 larger than w_1 was chosen in order to avoid a very small value of K_i.

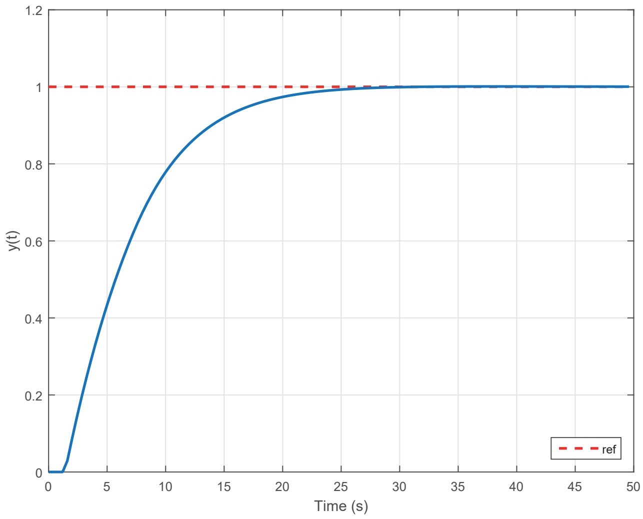

Fig. 5. Example 3: time response of small dead-time system.

The step response is shown in Fig. 5. One can see a small rise time and null overshoot.

The input signal is depicted in Fig. 6. One can see that the saturation allowance strategy works properly. No constraint on the states or output was violated.

Large dead-time

Consider a FOPDT system, borrowed from [13], with transfer function:

$$G(s) = \frac{5e^{-4s}}{2s+1} \tag{34}$$

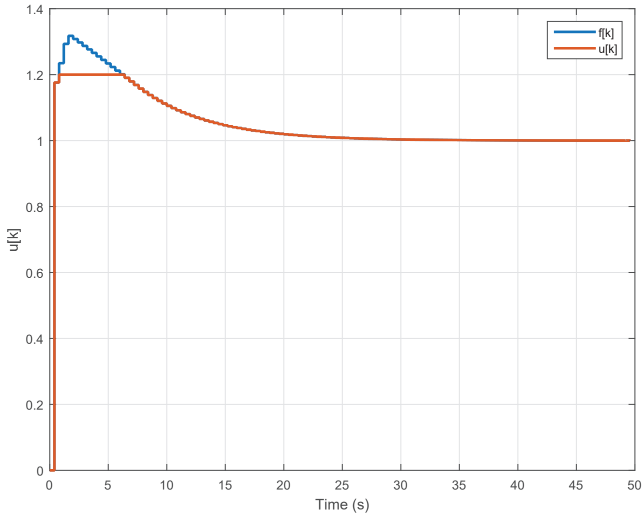

Fig. 6. Example 3: control signal for small dead-time system.

The sampling period was chosen as $T_s = 0.4\ s$. For the discretization, the zero-order holder method was used. The discrete-time transfer function is:

$$G(z) = \frac{0.9063}{z^{10}(z - 0.8187)} \tag{35}$$

A 11-th order state space realization in the observable canonical form was obtained. The system is subject to the following constraints:

$$\begin{array}{c} -10 \leq y[k] \leq 10 \\ -1.05 \leq u[k] \leq 1.05 \end{array} \tag{36}$$

and the additional constraints were set to:

$$\begin{array}{c} -10 \leq x_i[k] \leq 10,\ for\ i = 1, \cdots, 11; \\ -100 \leq v[k] \leq 100 \\ -10 \leq \omega[k] \leq 10 \end{array} \tag{37}$$

The input bounds were chosen in order to explore the potential of the proposed method. By applying the bilinear design strategy (24), the following results were obtained, with weighting factors w_1 = 3, w_2 = 4. The positively invariant polyhedron has 2970 vertices, obtained with a matrix $\bar{L}$ with 34 rows. The obtained controller gains were K_p = 0.06878, K_i = 0.02698, and $\delta = 3.0013 \times 10^{-8}$.

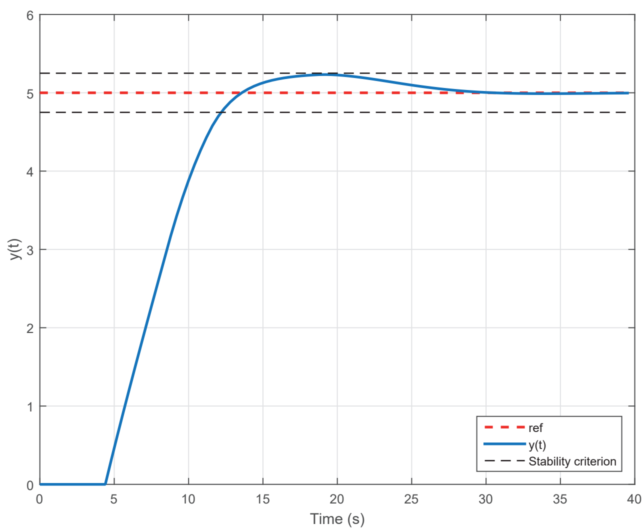

Fig. 7. Example 3: time response of a large dead-time system.

In Fig. 7, the system response is shown. One can notice that the settling time is quite large, but consistent with the large dead-time. There is a trade-off between the tracking performance and a non-oscillatory transient response. If the controller's gains are increased to obtain a smaller rise time, the response goes very oscillatory. Considering the 5 % settling time criterion ($t_{s5\%}$), the output controlled by PI proposed stabilized after 12 s.

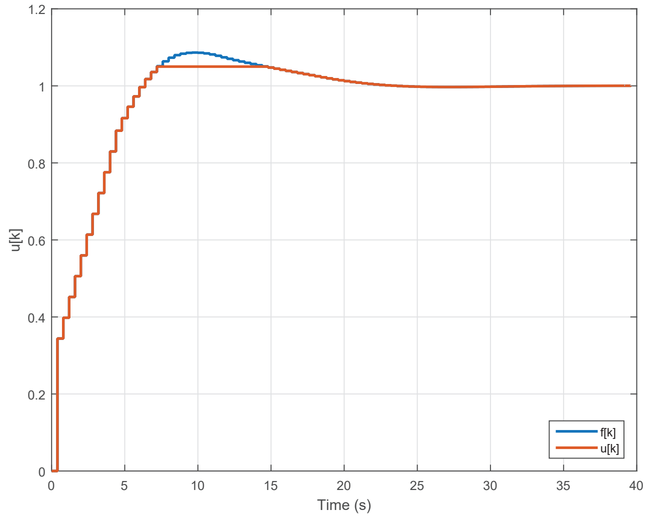

Fig. 8. Example 3: control signal of a large dead-time system.

The input signal is depicted in Fig. 8. Normally, saturation occurs when there is a change in the reference signal, but in this case, due to the high dead-time, saturation occurred later. The control signal saturated for 8 s which is quite a long time, but the proposed method delivered a controller which enforces the constraints.

Example 4: Open-Loop Unstable System

The main objective of this example is to show that the proposed method has the potential to find a PI tuning for open-loop unstable systems, and to compare the proposed method with another PI tuning method based on invariant sets and bilinear programming.

The tuning method chosen was that proposed in [17]. This method addresses constant reference tracking for SISO systems subject to state and control constraints via a PI control law with a feedforward term. It also guarantees local closed-loop stability and constraints fulfilment, using contractivity properties of polyhedral sets.

The approach is based on the positive invariance property applied to an extended state-space representation that includes a reference model, leading to more complex conditions than those based on the Δ-invariance property.

Consider the following linear time-invariant, open-loop unstable systems, borrowed from [17], represented by:

$$\begin{array}{c} \dot{x}(t) = 0.2x(t) + u(t), \\ y(t) = x(t), \end{array} \tag{38}$$

The sampling period was chosen as $T_s = 0.5\ s$ and zero-order holder method was used for discretization. A state-space realization is given by:

$$\begin{array}{c} x[k+1] = 1.1052x[k] + 0.5259u[k] \\ y[k] = x[k] \end{array} \tag{39}$$

The system is subject to the following constraints:

$$\begin{aligned} -3.33 \leq y[k] \leq 3.33 \\ -1 \leq u[k] \leq 1 \end{aligned} \tag{40}$$

and the additional constraints are set as:

$$\begin{aligned} -60 \leq v[k] \leq 60 \\ -2 \leq \omega[k] \leq 2 \end{aligned} \tag{41}$$

By applying the bilinear design strategy (24), the following results were obtained. We compared the closed-loop response obtained with the proposed PI to that obtained with the Δ-invariance approach of [17].

The number of rows of $\bar{L}$ was fixed to 6. Increasing the value of δ implies a slower speed of the system's response. Thus, three different lower bounds were imposed for the range of δ values as shown in Table 2. Therefore, this test makes it possible to show the different behaviors with significant changes in the variable δ.

Table 2 shows the controller parameters.

Range of δ	K_p	K_i	w_1	w_2	δ
$0 < \delta < 1$	2.3289	0.4499	1	1	8.0705×10^{-7}
$0.8 \leq \delta < 1$	2.1149	0.2297	1	0	0.80
$0.9 \leq \delta < 1$	2.3073	0.4073	1	0	0.90

Table 2. Controller parameters.

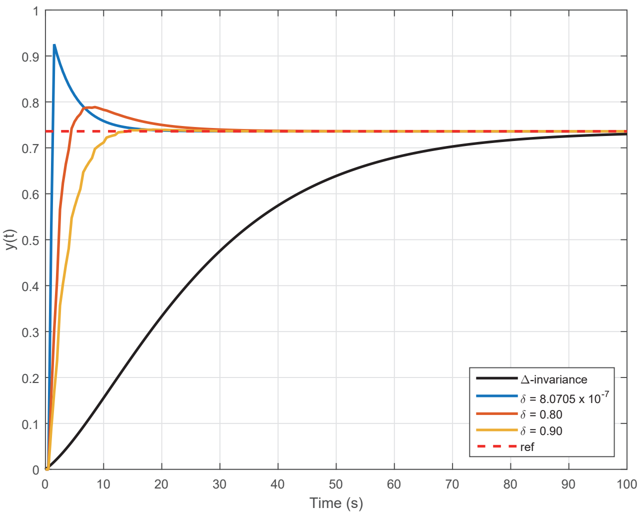

Fig. 9. Example 4: system response.

Table 3 shows the performance indexes that were compared considering 6 criteria: overshoot, settling time t_s, saturation interval, δ, IAE, and ITAE.

Controller	Δ-invariance	Proposed PI		
		1	2	3
Overshoot (%)	0	25.7	6.9	0
t_s (s)	100	18	40	20
Saturation interval (s)	0	0.4	0	0
δ	-	0.0	0.80	0.90
IAE	0.206	0.025	0.031	0.037
ITAE	4.147	0.044	0.102	0.076

Table 3. Comparison of the performance evaluation of the plant controlled by proposed PI and PI obtained using the Δ-invariance.

In Fig. 9, four different responses are depicted. One of them was obtained by the Δ-invariance method, and the other three were obtained by the proposed method. The difference between the responses of the proposed method is the change in the variable δ. It is possible to observe that the lowest δ value was associated to the larger overshoot. Saturation occurred when we had the largest error. In this situation, one of the PI controllers proposed saturated during 0.4 s, whereas others, including that obtained using Δ-invariance did not saturate. When we observe the settling time, the system controlled by the proposed PI had a faster response than that obtained using Δ-invariance. The controller performance indices only stresses the results shown in Fig. 9. Even for the worst case of IAE, the proposed controller showed an improvement of at least six times, and for the worst case of ITAE, a very significant improvement can also be seen. Thus, for the criteria evaluated, it was possible to observe that even for an open-loop unstable system, the proposed method resulted in a better response. Also, the designer can adjust the optimization parameters in order to shape the response as desired.

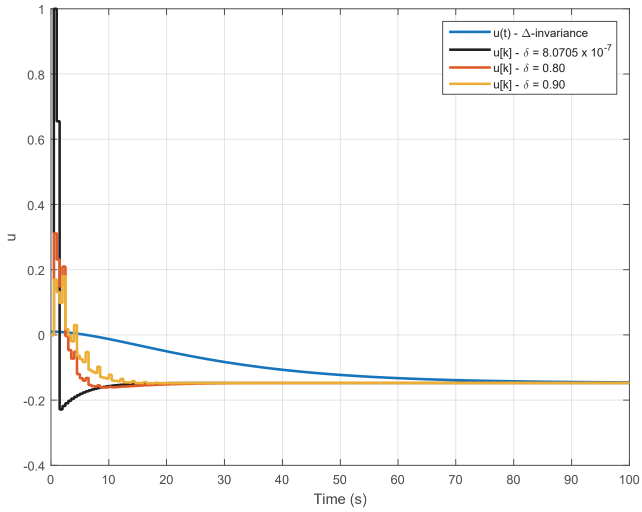

Fig. 10. Example 4: control signals.

Fig. 10 shows the control signal. The control input saturated for a short time.

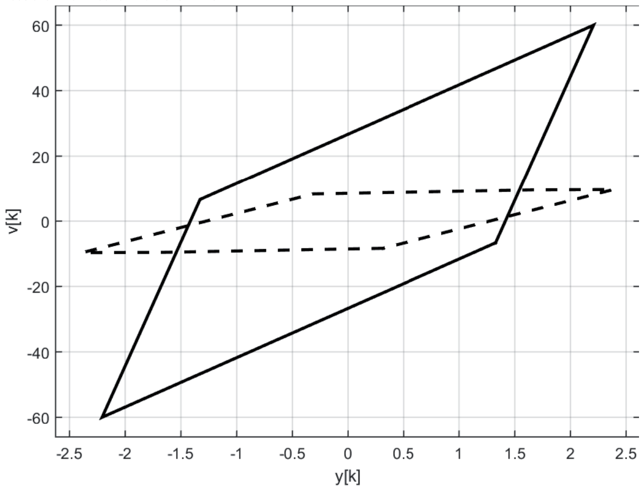

Fig. 11. Example 4: comparison of polyhedron sizes. Δ-invariance approach (solid black line) and proposed approach with δ = 0.80 (dashed black line).

The positively invariant set stands for the set of admissible initial states, i.e. the initial states for which constraint satisfaction is guaranteed. It is important to have it as large as possible. In Fig. 11 the sets obtained by the proposed method with $\delta = 0.8$ and by the Δ-invariance approach are depicted. Even though this later is larger, one can see that it admits larger values of $v[k]$, the integral of the error, which, in general, is not a variable of interest regarding their initial values. On the other hand, the polyhedron obtained with the proposed approach admits a larger interval for the initial output values (2.38 vs 2.20).

Conclusion

An optimization-based method for tuning standard PI controllers for linear systems of any order subject to constraints on the controlled variable and actuator saturation was presented in this work. The strategy used to deal with the constraints was based on invariant polyhedral sets, and polytopic modelling was used to deal with saturation allowance. The conditions for solution were transformed into an optimization problem with bilinear constraints, whose solution delivers an invariant set complying with the constraints and the associated PI controller gains. A state space formulation of the closed-loop system was developed, which includes a filter for the reference signal that provides more degrees of freedom to the optimization problem.

To illustrate the method, a comparison with other PI controller tuning methods was performed through numerical experiments with open-loop stable and unstable models. The potential of the method applied to systems with small and large dead-time was also explored. We also illustrated how the parameters of the objective function can be chosen in order to obtain the desired behaviour of the output response. For systems with dead-time, the method worked properly without the need of additional modifications or constraints.

The proposed method resulted in controllers with better performance when compared to other methods in the literature. In Example 1, our controller resulted in a 32 % smaller IAE and a 50 % smaller saturation interval when compared with that of reference [15]. Furthermore, it did not show overshoot, against 5.5 % of that in [15]. In example 4, our controller resulted in a 60 % smaller settling time, a 98 % smaller ITAE, and a 85 % smaller IAE than those in reference [17].

In future work, extension of the method to linear and nonlinear PID controllers shall be investigated.

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