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doi:10.15199/48.2022.12.30

On spectrum of Metzler matrices

Abstract. *In the paper it was proven that spectrum of Metzler matrix must belong to a certain cone in the complex plane. The result is derived from the analytical characterization of spectra of positive matrices obtained by Karpelevich. Furthermore, it was shown that in case of 3x3 matrices this property yields also a sufficient condition that a set of numbers must satisfy in order to be spectrum of some Metzler matrix.*

Streszczenie. *W pracy wykazano, że widmo macierzy każdej Metzlera należy do pewnego stożka na płaszczyźnie zespolonej. Wykorzystano w tym celu analityczną charakteryzację widma macierzy dodatniej wyznaczoną przez Karpielewicza. Ponadto wykazano, że w przypadku macierzy 3x3 własność ta pozwala wyznaczyć również warunek wystarczający, który spełniać musi zbiór liczb zespolonych, aby był widmem pewnej macierzy Metzlera.* ***(O widmie macierzy Metzlera)***

Keywords: Metzler matrix, spectrum, eigenvalues, Karpelevich region, positive system
Słowa kluczowe: macierz Metzlera, widmo, wartość własna, region Karpielewicza, system dodatni

Introduction

In recent years positive dynamical systems have attracted a remarkable interest among researchers. Likewise, Metzler matrices, closely related to continuous-time linear dynamical systems. It is known (see e.g. [1]) that spectrum of Metzler matrix has a special form, namely, it is equal to spectrum of some non-negative matrix shifted by some real number. Perron proved that positive matrix has a real positive eigenvalue equal to its spectral radius [2]. Frobenius generalized his results to non-negative matrices [3]. Karpelevich showed in [4] that spectrum of non-negative matrix belongs to a certain region (so-called Karpelevich region) of the complex plane; his results were later extended by Benvenuti and Farina in [5]. In the following paper Karpelevich theory is used to show that the spectrum of Metzler matrix must belong to a certain cone in the complex plane. Moreover, it is proven that in case of 3×3 matrices, this is provides not only necessary, but also sufficient condition for spectrum of Metzler matrix.

In the sequel, all matrix and vector inequalities shall be considered component-wise. For function f defined for $t \geq 0$, notation $f \geq 0$ means: $\forall t \geq 0 \quad f(t) \geq 0$. $\sigma(A)$, $\rho(A)$ and $\alpha(A)$ denote the spectrum, the spectral radius and the growth constant (i.e. the maximum of eigenvalues real-parts) of matrix A, respectively. For point x and set Y, by $x + Y$ we denote the set $\{x + y : y \in Y\}$.

Metzler matrices

First, let us recall basic definitions and facts regarding Metzler matrices and positive systems.

Definition 1. *Matrix is called a Metzler matrix if all its off-diagonal entries are non-negative.*

Metzler matrices are closely related with positive linear time-invariant systems. In order to demonstrate that, let us consider a dynamical system:

(1)
$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) \\ &x(0) = x_0 \\ &x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m} \end{aligned}$$

Definition 2. *System (1) is positive if*

(2)
$$\forall x_0 \geq 0, u \geq 0 : \; x \geq 0$$

Lemma 1. *System (1) is positive if and only if*

(3)
$$B \geq 0 \;\; \wedge \;\; \forall t \geq 0 : \; e^{At} \geq 0$$

Lemma 2. *For matrix A the following properties are equivalent [6]:*

(i) $\forall t \geq 0 : \; e^{At} \geq 0$

(ii) A *is a Metzler matrix*

Spectrum of Metzler matrices

It can be seen easily that each Metzler matrix M is equal to a sum of some non-negative matrix N and the identity matrix scaled by some real factor η (where η can be any number greater or equal to the least entry of the diagonal of M). Consequently, the spectrum of M is the spectrum of N shifted by η [1]. On account of that, spectral properties of Metzler matrices can be investigated through analysis of spectra of non-negative matrices. By well-known Perron-Frobenius theorem, N has an eigenvalue equal to its spectral radius $\rho(N)$. Furthermore, this eigenvalue is strictly greater than any other eigenvalue of N in terms of real part. Since spectrum translation preserves the latter property, as a corollary, we can state the following necessary condition on spectrum of Metzler matrix [7]:

Condition 1. *Metzler matrix M has a real eigenvalue λ_{max} such that:*

$$\forall \lambda \in \sigma(M) \setminus \{\lambda_{max}\} : \; Re(\lambda) < \lambda_{max}$$

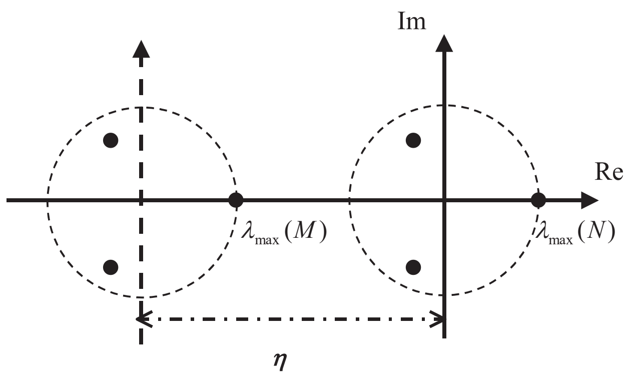

Fig. 1. Spectra of Metzler and non-negative matrices, respectively

Let us now consider a set of eigenvalues of non-negative matrices with given spectral radius, i.e. a set $\theta_n^r = \{\lambda \in \mathbb{C} : \exists N \in \mathbb{R}^{n\times n} \; \lambda \in \sigma(N), N \geq 0, \rho(N) = r\}$. It is easy to notice that θ_n^r can be obtained by scaling θ_n^1, which in turn satisfies:

Theorem 1. *θ_n^1 is characterized as follows (proven originally in [4]; statement later simplified in [8]):*

a) θ_n^1 is contained in the unit disc of the complex plane and is symmetric with respect to the real axis,

b) θ_n^1 intersects the unit circle in a finite number of vertices given by $\nu_{a,b} = e^{2\pi i a/b}$, for $a, b \in \mathbb{N}, 0 \leq a < b \leq n$,

c) the arc between consecutive vertices ν_{a_1,b_1},ν_{a_2,b_2} for $b_1 \leq b_2$ where a_1, b_1 and a_2, b_2 are pairwise co-prime, is *included in the set of points λ satisfying parametric equation:*

$$\lambda^{b_2}(\lambda^{b_1} - s)^{[n/b_1]} = (1-s)^{[n/b_1]}\lambda^{b_1[n/b_1]} \quad (4)$$

where s runs over the interval $0 \le s \le 1$.

Set θ_n^r is referred to as Karpelevich region. Please note that by Perron-Frobenius theorem, the rightmost vertex of θ_n^r, i.e. $\nu_{0,1} = r$, is equal to the growth factor of each non-negative matrix with spectral radius r.

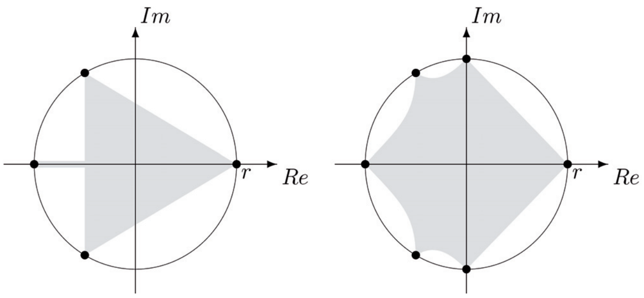

Fig. 2. Regions θ_3^r and θ_4^r (for analytical description please refer to [9])

For $n > 2$ region θ_n^r has two properties which will be useful in further reasoning:

Lemma 3. *The arcs of θ_n^r joining the rightmost vertex $\nu_{0,1}$ with its adjacent vertices $\nu_{1,n}$, $\nu_{n-1,n}$ are straight line segments with slopes $\pm\frac{\sin(2\pi/n)}{\cos(2\pi/n)-1}$.*

Proof. It suffices to prove the lemma for θ_n^1. It follows from (4) that the considered arcs consist of points satisfying:

$$(\lambda - s)^n = (1-s)^n \quad (5)$$

which for $s \neq 1$ is equivalent to:

$$\left(\frac{\lambda - 1}{1-s} + 1\right)^n = 1 \quad (6)$$

For $s = 0$ and for $s = 1$ equation (5) yields $\lambda^n = 1$ and $\lambda = 1$, respectively. On the other hand, from (6) it can be seen easily that changing s is equivalent to scaling $\lambda - 1$ by a real factor. Hence, the set of numbers λ satisfying (5) is a union of $n-1$ straight line segments connecting 1 with $e^{2\pi ik/n}$ for $k = 1, \ldots, n-1$ and the arcs must be straight line segments as well. Their slopes can be calculated easily from the coordinates of $\nu_{0,1}$, $\nu_{1,n}$ and $\nu_{n-1,n}$. ☐

Lemma 4. *Set θ_n^r is contained in a cone in the complex plane given by:*

$$\mu_n^r = \{\lambda \in \mathbb{C} : |Im(\lambda)| \le \frac{\sin(2\pi/n)}{\cos(2\pi/n)-1} Re(\lambda - r)\} \quad (7)$$

Proof. The inequality in (7) is equivalent to:

$$\frac{-\sin(2\pi/n)}{\cos(2\pi/n)-1}\mathrm{Re}(\lambda - r) \le \mathrm{Im}(\lambda) \le \le \frac{\sin(2\pi/n)}{\cos(2\pi/n)-1}\mathrm{Re}(\lambda - r) \quad (8)$$

Furthermore, let us notice that $\frac{\sin(2\pi/n)}{\cos(2\pi/n)-1} < 0$, so for λ such that $\mathrm{Re}(\lambda) > r$, the right hand side of the inequality in (7) is strictly negative, hence the inequality is not satisfied and $\lambda \notin \mu_n^r$. Equation (7) indeed describes a cone facing towards the left, symmetric with respect to the real axis, with its vertex in r. Let us choose $\lambda \in \mu_n^r$; we shall show that λ satisfies (8). If $\mathrm{Re}(\lambda) \ge r\cos(2\pi/n)$, then the projection of λ on the real axis lies between the projections of $\nu_{0,1}$, $\nu_{1,n}$ and $\nu_{n-1,n}$, and as a consequence λ must lie between the arcs connecting $\nu_{0,1}$, $\nu_{1,n}$ and $\nu_{n-1,n}$. Hence, (8) is satisfied by virtue of Lemma 3. If $\mathrm{Re}(\lambda) < r\cos(2\pi/n)$, then the inequality in (7) must be satisfied since $|\lambda| \le r$. The latter can be proved by observing that lines given by $\mathrm{Im}(\lambda) = \pm\frac{\sin(2\pi/n)}{\cos(2\pi/n)-1}\mathrm{Re}(\lambda - r)$ are secants of the circle $|\lambda| = r$, intersecting it in points r and $r(\cos(2\pi/n) \pm i\sin(2\pi/n))$. As a consequence, the points of these lines whose real part is strictly less than $r\cos(2\pi/n)$ lie outside of the circle and finally the set $\{\lambda : |\lambda| \le r \wedge \mathrm{Re}(\lambda) < r\cos(2\pi/n)\}$ must lie between these lines. ☐

The above-mentioned theorem and lemmas can be used to derive properties of spectrum of a Metzler matrix. Let $M = N + \eta I$, $M, N \in \mathbb{R}^{n\times n}$, $N \ge 0$. Then:

$$\sigma(M) = \eta + \sigma(N) \quad (9)$$

It follows directly from the definition of $\theta_n^{\cdot}$, that the spectrum of N satisfies:

$$\sigma(N) \subset \theta_n^{\rho(N)} \quad (10)$$

Hence:

$$\sigma(M) \subset \eta + \theta_n^{\rho(N)} \quad (11)$$

Taking into account the equality:

$$\alpha(M) = \alpha(N) + \eta = \rho(N) + \eta \quad (12)$$

one can obtain:

$$\sigma(M) \subset \alpha(M) - \rho(N) + \theta_n^{\rho(N)} \quad (13)$$

and finally:

$$\sigma(M) \subset \alpha(M) + \bigcup_{r\ge 0}(-r + \theta_n^r) \quad (14)$$

Using this result and the thesis of Lemma 4, for $n \ge 2$ we can obtain the main result of this paper, presented in the following:

Theorem 2. *The eigenvalues of Metzler matrix $M \in \mathbb{R}^{n\times n}$ with growth constant $\alpha(M)$ belong to the set:*

(15)

$$\mu_n^{\alpha(M)} = \{\lambda \in \mathbb{C} : |Im(\lambda)| \le \frac{\sin(2\pi/n)}{\cos(2\pi/n)-1} Re(\lambda - \alpha(M))\}$$

Proof. It follows directly from the definition of $\mu_n^{\cdot}$ that for $a, b \in \mathbb{R}$, $a + \mu_n^b = \mu_n^{a+b}$. Let us now choose $r \ge 0$. By Lemma 4, $\theta_n^r \subset \mu_n^r$, hence $(\alpha(M) - r + \theta_n^r) \subset (\alpha(M) - r + \mu_n^r) = \mu_n^{\alpha(M)}$. Finally, by (14), $\sigma(M) \subset \mu_n^{\alpha(M)}$. ☐

Example: Let us consider a dynamical system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (16)$$

with $A = \begin{bmatrix} -1 & 7 & 3 & 0 \\ 0 & -4 & -1 & -1 \\ 0 & 14 & 5 & -13 \\ 0 & 6 & 3 & -9 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

We want to determine if there exists a state space isomorphism P, such that the system with matrices PAP^{-1} and PB is positive. A necessary condition for this system to be positive is that the state matrix PAP^{-1} is a Metzler matrix. Let us examine the spectrum of PAP^{-1}: $\sigma(PAP^{-1}) = \sigma(A) = \{-1, -2, -3+3i, -3-3i\}$. PAP^{-1} satisfies

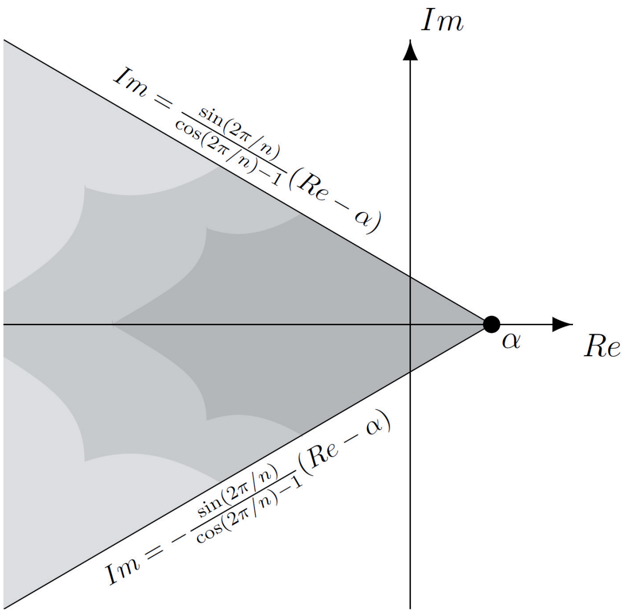

Fig. 3. Set μ_n^α (here for $n = 4$) containing eigenvalues of all $n \times n$ Metzler matrices with fixed growth constant α. Shaded regions are shifted Karpelevich regions $(\alpha - r + \theta_n^r)$

Condition 1 and its largest real-part eigenvalue $\alpha(PAP^{-1})$ is equal to -1. Hence, the inequality in (15) has a form $|\mathrm{Im}(\lambda)| \leq \mathrm{Re}(\lambda + 1)$ and it is not satisfied by eigenvalues $-3 \pm 3i$. As a consequence, the spectrum of PAP^{-1} does not satisfy the thesis of Theorem 2 and PAP^{-1} cannot be a Metzler matrix. Finally we can conclude that there does not exist a coordinate change P which would transform (16) into a positive system.

Lemma 5. *Each set Σ consisting of at most 3 complex numbers, symmetric with respect to the real axis, satisfying Condition 1 and the thesis of Theorem 2, is a spectrum of a 3×3 Metzler matrix.*

Proof. If $\Sigma \subset \mathbb{R}$, one can construct a Jordan matrix whose spectrum is equal to Σ and which is a Metzler matrix. On the other hand, if Σ contains nonreal numbers, it must have a form:

$$\Sigma = \{\alpha, \beta + \gamma i, \beta - \gamma i\}, \alpha, \beta, \gamma \in \mathbb{R}, \alpha > \beta, \gamma > 0 \tag{17}$$

Since Σ satisfies (15), every number $\lambda \in \Sigma$ must satisfy:

$$|\mathrm{Im}(\lambda)| \leq \frac{-1}{\sqrt{3}}\mathrm{Re}(\lambda - \alpha) \tag{18}$$

which for $\lambda = \beta \pm \gamma i$ yields:

$$\sqrt{3}\gamma \leq \alpha - \beta \tag{19}$$

For

$$J = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & -\gamma & \beta \end{bmatrix}, P = \begin{bmatrix} 1 & -1 & \sqrt{3} \\ 1 & -1 & -\sqrt{3} \\ 1 & 2 & 0 \end{bmatrix}$$

one obtains:

$$M = PJP^{-1} = \frac{1}{3}\begin{bmatrix} \alpha + 2\beta & \alpha - \beta + \sqrt{3}\gamma & \alpha - \beta - \sqrt{3}\gamma \\ \alpha - \beta - \sqrt{3}\gamma & \alpha + 2\beta & \alpha - \beta + \sqrt{3}\gamma \\ \alpha - \beta + \sqrt{3}\gamma & \alpha - \beta - \sqrt{3}\gamma & \alpha + 2\beta \end{bmatrix}$$

which by (19) is a Metzler matrix. □

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