1. Karol BOLEK[1], 2. Michał K. URBAŃSKI[1]

Warsaw University of Technology (1)
ORCID: 1. 0000-0002-0995-7760; 2. 0000-0001-9429-1555

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Influence of Measuring System Noise on the Fractal Dimension of the Chaotic Signal Attractor

***Abstract**. The present paper investigates the influence of stochastic noise on the estimation of the fractal dimension of the chaotic signal attractor for additive and multiplicative noises in the frequency and time domain. A simplified analogue measuring system noise model was proposed as the amplitude and phase noise in the Fourier domain, which was the equivalent of multiplicative and additive noise in the time domain. A numerical experiment was performed, which introduced noise of various intensities into the chaotic signal from the Chua system. It has been shown that in the logarithmic diagram of the correlation integral, additional scaling regions appear, the range of which increases with increasing noise intensity, causing dimension estimation errors. It has also been shown that without a thorough analysis of the correlation integral, deterministic noise can be easily confused with stochastic noise in the frequency domain.*

***Streszczenie.** W pracy przedstawiono wpływ szumu stochastycznego na estymacje wymiaru fraktalnego atraktora sygnału chaotycznego dla szumów addytywnych i multiplikatywnych w dziedzinie częstości i czasu. Zaproponowano uproszczony model szumów analogowego toru pomiarowego jako szum amplitudowy i fazowy w dziedzinie Fouriera, stanowiący odpowiednik szumu multiplikatywnego oraz addytywnego w dziedzinie czasu. Wykonano eksperyment numeryczny, za którego pomocą wprowadzano do sygnału chaotycznego pochodzącego z numerycznego układu Chua szum o różnych natężeniach. Pokazano, że na wykresie logarytmicznym całki korelacyjnej pojawia się dodatkowe obszary skalowania, których zakres rośnie wraz ze wzrostem natężenia szumów, powodując błędy estymacji wymiaru. Pokazano również, że bez wnikliwej analizy całki korelacyjnej szum deterministyczny łatwo pomylić z szumem stochastycznym w dziedzinie częstości. (**Wpływ szumu stochastycznego na estymacje wymiaru fraktalnego atraktora sygnału chaotycznego dla szumów w dziedzinie częstości i czasu**)*

Keywords: fractal dimension, chaotic signal, measuring system noise, phase noise.
Słowa kluczowe: wymiar fraktalny, sygnał chaotyczny, szum układu pomiarowego, szum fazowy.

Introduction

Chaotic systems have been the object of interest of many researchers for years. More and more often, chaotic behaviour is discovered in systems already well known to science, showing new aspects of the problem. The analysis of deterministic systems provides a wide range of new tools. Complexity measures such as Lyapunov exponent, metric and topological entropy, and fractal dimension are invariant concerning diffeomorphisms, which allows characterizing a system with many degrees of freedom through the observation of only a single variable [1]. However, each measurement signal is exposed to noise, which may affect further signal analysis. Careless determination of signal parameters in the presence of noise may result in incorrect estimation or even incorrect classification of the stochastic system as deterministic [2,3].

In works [4-9] it was shown that white noise (uncorrelated) causes an increase in the attractor fractal dimension and the largest Lyapunov exponent. Ben-Mizrachi et al. [10] pointed out that when the fractal dimension estimation is determined using the popular Grassberger method [11,12], then two scaling areas in the diagram of the correlation integral are observed: one coming from noise (for small scales) and the other coming from the attractor (for large scales). In the works [6,13,14, 15] this information was used to develop a method of removing noise from the signal and improving the dimension estimation. In [7,15], the application of the Gaussian kernel to the correlation integral was proposed, which improved the smoothness of the scaling areas. Zuo et al. [8] compared the effect of noise on fractal structures of different dimensions. It turns out that noise has a stronger impact on low-dimensional structures than on high-dimensional structures. The authors explain this by the fact that low-dimensional structures are more "rough", and the noise smoothes them, making them more similar to high-dimensional ones. It is also worth adding that for stochastic noise, the fractal dimension cannot be determined because it fills the entire phase space without saturating as a function of the embedding dimension. Proof of this claim can be found in Jayawardena et al. [9].

In order to eliminate the influence of noise, many authors use various filtering techniques. However, filtering has a great influence on the signal values and can significantly distort the estimation of the dimension. In works [16,17,18] an increase in the correlation dimension was noticed as a result of the use of a simple RC low-pass filter. The explanation for this phenomenon has been associated by Badii et al. [19] with the appearance of an additional Lyapunov coefficient, which changes the attractor dimension. Mitschke [20] claims that acausal filters cause deep phase shifts as a function of frequency, but when the filter phase is constant, the filtering does not change the dimension. Moreover, in [21], the authors show that strong filtering of the noise signal may result in obtaining a finite value of the correlation dimension (the integral becomes saturated).

Argyris et al. in the works [4,5], following the example presented in [22], divided noises into two groups: dynamic noises, i.e. those that affect the evolution of the system, and the output noise, i.e. those that appear directly in the measurement signal. In addition, they distinguished how noise can appear in the signal: additively or multiplicatively.

Most of the works discussed consider noise in a signal as dynamic and additive noise as a function of time. However, this is not the only possibility of noise. From the metrological perspective, the noise of the measuring system is more interesting, which can be both additive and multiplicative noise. In particular, the last one will be associated with the negative feedback loop present in amplification circuits. Moreover, a deterministic signal as a time-varying is characterized not only by amplitude but also by phase. In such a case, the analysis should be extended also to include the influence of the phase angle noise. The phase versus time of chaotic signals is not well defined due to the non-stationary nature of the signal. However, in the frequency domain, using the Fourier transform makes it possible to separate the amplitude and phase components. This approach can much better represent the instabilities of the dynamic parameters of the amplification circuits. In addition, the influence of the filter phase spectrum on the correlation dimension indicated by Mitschke [20] becomes a justified premise to supplement the existing research.

This paper presents the influence of stochastic noise on the estimation of the attractor correlation dimension coming from the Chua system. First, it is shown how the dimension changes with increasing noise output as a function of time. The noise was added to the signal amplitude as an additive and multiplicative. These results confirm the conclusions of the literature research. An analogous analysis was then performed by adding noise to the amplitude and phase parts of the Fourier transform of the signal.

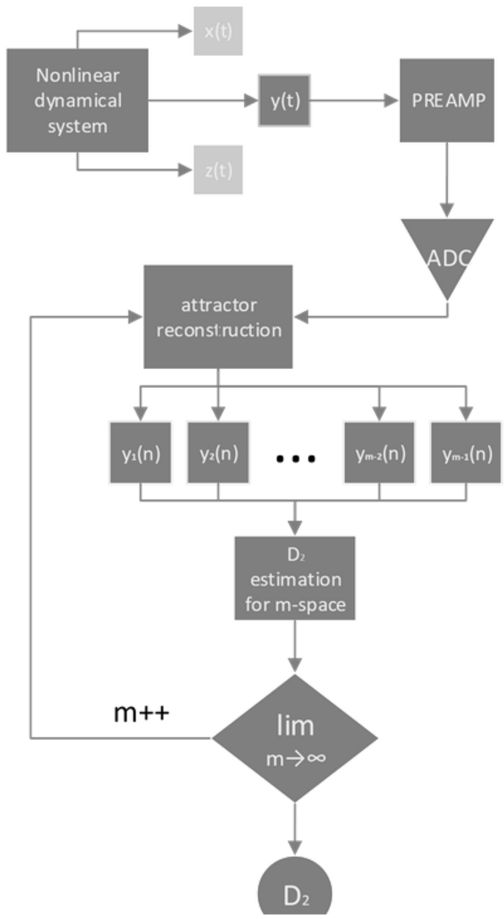

Fig.1. Diagram of the fractal dimension measurement procedure. I) Data acquisition – the continuous signal $y(t)$ coming from the dynamic system is preamplified and filtered by the preamplifier stage, then converted to a discrete signal y_n by an analog-to-digital converter (ADC), II) Reconstruction of the phase space – because only single-variable is measured, the trajectory in phase space should be reconstructed in such away as to retains the properties of the dynamical system, III) Dimension estimation – the reconstructed signal is given a numerical analysis, which consists in calculating the sum of C(r), and then determining the slope in the region of linear scaling in the logarithmic plot of C(r) for the m-dimensional embedded space.

Structure of the experimental system

On the basis of Takens' theorem [1], on the reconstruction of the topological properties of the attractor of a dynamic system, it is enough to observe the one-dimensional signal generated by this system. A diagram of a typical fractal dimension measuring system is shown in Figure 1. The measurement can be divided into three stages: I) data acquisition, II) phase space reconstruction and III) dimension estimation.

In the first step, the source of the signal of the dynamic system $X(t)$ is determined, and then it is registered over a certain time interval, obtaining a series of discrete samples $x = \{x_1, x_2, \cdots, x_N\}$. In the next stage, the phase space is reconstructed using the original measurement series, constructing m-dimensional vectors whose components are samples of the measured signal delayed successively by time τ (expressed in sampling interval T units) [1]. The resulting trajectory is described by the following vector:

(1) $$x_i = \left(x_i, x_{i+\tau}, \cdots, x_{i+(m-1)\tau}\right)$$

where m is the dimension of the embedded space of the reconstructed attractor. The correlation dimension can be determined using the Grassberger method [11,12]. It consists in determining the sum of the number of points located inside the m-dimensional hypersphere with radius r and the centre at each of the following trajectory points. This sum is given by the following function:

(2) $$C_m(r) = \frac{2}{N_m(N_{m-1})} \sum_{i<j} \theta\left(r - \|x_i - x_j\|\right)$$

where $\theta(x)$ is the Heaviside function with $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$ and $N_m = N - (m-1)\tau$. The slope of a linear area in a logarithmic plot is called the scaling exponent. Stages II) and III) are repeated until the exponent stops growing with increasing the embedded space. Ultimately, the saturation value is called the correlation dimension.

(3) $$D_2 = lim_{m\to\infty} \frac{\log C(r)}{\log r}$$

The main source of noise in the measurement signal is the analogue circuit. In order to match the signal to the bandwidth and dynamics of the ADC, it must be amplified and filtered out (Fig. 2). The real systems that perform these operations will always be sources of noise. If we are able to describe the noise with an equivalent current (or voltage) source, then such a source appearing in the signal chain will be an algebraic sum with the measurement signal (additive noise). If, on the other hand, the noise sources are in the feedback loop circuit, such noise will be called multiplicative noise.

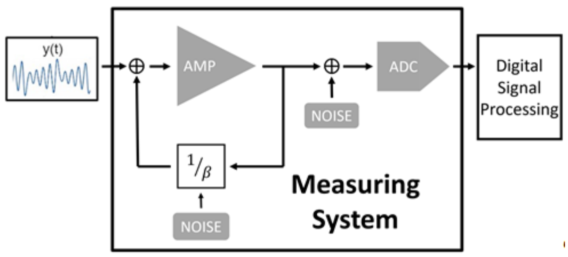

Fig.2. The diagram shows a simplified noise model of the analog measurement system. Depending on the line in which the noise appears, we can distinguish additive noise (in the signal processing line) or multiplicative noise (in the feedback loop).

Results

A numerical experiment was performed to simulate the conditions of introducing noise of controlled intensity into a raw signal. The numerical solution of the Chua system [23] was used as the measured signal. This system is described by three first-order differential equations. The solution of these equations generates a double-scrolled attractor with a correlation dimension of 2.05. The reconstruction of dimension was analysed with additive and multiplicative noise disturbance in the time domain and with the amplitude (multiplicative) and phase additive noise in the frequency domain as shown in Table 1. The noise $w = \{w_1, \cdots, w_N\}$ came from the Gaussian distribution with parameters $(0, \sigma)$ for additive noises and $(1, \sigma)$ for multiplicative noises.

Table 1. Methods of introducing noise into the measurement signal

	additive	multiplicative
In time	$\tilde{x}_n = x_n + w_n$	$\tilde{x}_n = x_n \cdot w_n$
In frequency	$\tilde{y}_n = \mathcal{F}^{-1}(Y_n \cdot \exp(-iw_n))$	$\tilde{y}_n = \mathcal{F}^{-1}(Y_n \cdot \lvert w_n \rvert)$

In the time domain, the additive noise is understood as the sum of each sample x_n of the measurement series with the corresponding value of the random variable w_n. Multiplicative noise is the product of each of the signal samples with the corresponding noise sample. The measure of noise intensity is the standard deviation of the distribution from which the noise was taken in relation to the standard deviation of the raw signal. The correlation dimension was determined in six test series, in a 10-dimensional embedded space, for noise intensities in the range 10^{-4} to 10^{-1}. The results are presented in Figure 3. An important conclusion from the experiment is the doubtless increase in the dimension with the increase in noise. The dimension increases faster for additive noise than for multiplicative noise. Not two (as expected) but at least three regions of linear scaling were obtained in the logarithmic plot of the function C(r). The region associated with the attractor assumes a constant value around 2 and occurs for large radii. In the area associated with noise (for small radii), the dimension increases consistently. The region marked in black on the plot is the linear regression area that was used in the further estimation process. Scaling regions with a very low slope (~1) are for very large radii, and its interpretation is unknown to the authors.

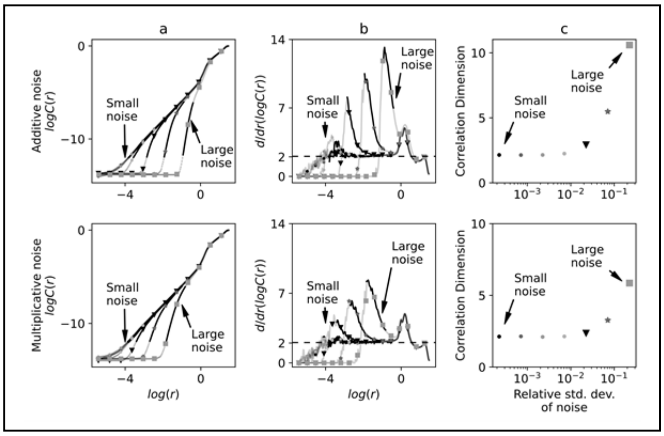

Fig.3. Results of the noise impact on the chaotic signal as a function of time a) correlation integrals for m = 10, b) slope of correlation integrals, c) correlation dimension for successive values of noise intensity. Standard deviation was related to the deviation of the analyzed signal. The black area shows the range of the linear regression.

The frequency domain analysis consisted in performing the Fourier transform of the test signal. The analytical form of the signal allows for the separation of the amplitude part A_n and ϕ_n phase in the following form $X_n = A_n exp(-i\phi_n)$. Thus, the multiplication of the data prepared in this way by the noise constant w_n and the execution of the inverse transform gives the desired signal. According to the classification given in Table 1, the noise amplitude w_n is the multiplicative component of the signal amplitude and the phase angle is the additive component of the signal phase. Figure 4 shows the results of dimension estimation for noise with an absolute deviation in the range of 10^{-3} to 1 for amplitude noise, and 10^{-3} to 10^{-1} with respect to 2π for phase noise. As in the previous case, the dimension increases with increasing noise for both: amplitude and phase noise. Also, as before, there are at least two scaling regions, one of which corresponds to noise (small scales) and the other to an attractor (large scales). This time, the low-dimensional scaling region occurs largely from small radii and appears for high-intensity noise. In addition, Figure 5 shows the mixed dependence of the occurrence of amplitude and phase noise at the same time of different intensity on the scales as in the previous one.

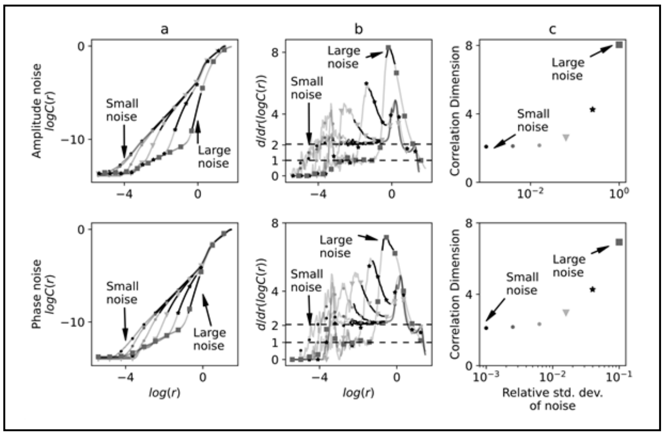

Fig.4. Results of the influence of amplitude and phase noise as a function of frequency on the chaotic signal a) correlation integrals for m = 10, b) slope of correlation integrals, c) correlation dimension for successive values of noise intensity. The standard deviation of the noise for the phase was related to 2π, for the amplitude is given in absolute terms.

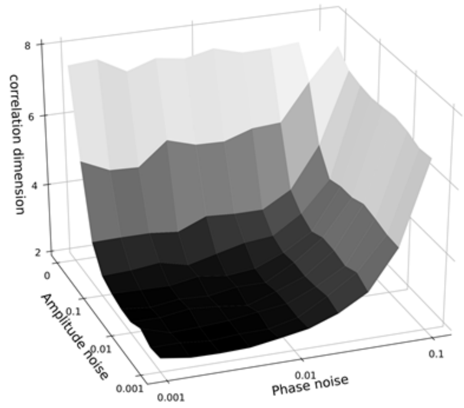

Fig.5. A plane showing the joint effect of phase and amplitude noise on the measurement of the fractal dimension.

Conclusion

The results obtained for the noise in the time domain confirm the previous literature reports. As it can be seen, it is quite easy to make a mistake in determining the correlation dimension without a detailed analysis of individual region of the integrals. The increase in dimension is the result of a wrong scaling region selection, which includes both noise and attractor regions. As the noise level increases, the intersection of the regions shifts towards larger radii. Thus, the observed exponent is the resultant of the other two.

The noise analysis in the frequency domain showed that the increase in the noise intensity causes the same effect as in the time domain. A novelty is the appearance of an additional low-dimensional (~1) scaling area for the small radii, which with increasing noise shifts the new intersection point towards the large radii. It is difficult to explain why this effect occurs, but it can undoubtedly be another source of dimension estimation error that did not occur in the case of time analysis. Another important observation is that while broadband time-domain noise is generally fairly easy to distinguish from deterministic noise, a frequency-domain distortion is not. Such a distorted signal may still resemble a deterministic signal both in the time and Fourier domain (Fig. 6).

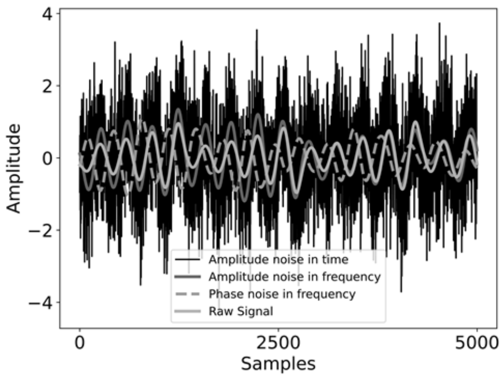

Fig.6. Comparison of amplitude noise in the time and frequency domains and phase noise in the frequency domain with a chaotic signal as a function of time. The noise intensity values were selected in such a way that the signal could be considered random.

The connection of noise in the frequency domain as instability of dynamic parameters of measuring systems is a new element of the analysis of the influence of random noise on the reproduction of deterministic chaos complexity measures. As the authors were concerned with the metrological approach to the problem, this publication covers only the input noise analysis. For all research on chaotic systems, however, it would be advisable to extend this analysis also to dynamic noise.

***Authors**: mgr inż. Karol Bolek, E-mail: karol.bolek.dokt@pw.edu.pl, dr hab. inż. Michał K. Urbański, E-mail michal.urbanski@pw.edu.pl, Faculty of Physics, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw*

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