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Sliding mode control to stabilization of nonlinear Underactuated mechanical systems

***Abstract**. In this paper, a sliding mode control was applied to a particular underactuated mechanical system, the* inertia wheel inverted pendulum. *This mechanical system presents strong non-linearities and instabilities in its dynamic modeling. The technique found for control by sliding mode is very easy to implement, it presents a simple control algorithm which can be easily implemented in a real time system. The simulation results obtained are very interesting, they show the efficiency of the proposed approach and that it has good performance in terms of robustness and stability of convergence both for stabilization and for the rejection of external point disturbances to the system.*

***Streszczenie.** W tym artykule, sterowanie trybem ślizgowym zostało zastosowane do konkretnego niedostatecznie uruchamianego układu mechanicznego, odwróconego wahadła koła bezwładności. Ten układ mechaniczny wykazuje silne nieliniowości i niestabilności w modelowaniu dynamicznym. Technika sterowania w trybie przesuwnym jest bardzo łatwa do wdrożenia, przedstawia prosty algorytm sterowania, który można łatwo zaimplementować w systemie czasu rzeczywistego. Uzyskane wyniki symulacji są interesujące, pokazują skuteczność proponowanego podejścia oraz dobre wyniki w zakresie odporności i stabilności zbieżności zarówno dla stabilizacji, jak i dla zewnętrznych zakłóceń punktowych do systemu. (**Sterowanie w trybie ślizgowym do stabilizacji nieliniowych układów mechanicznych o niedostatecznym uruchomieniu**)*

Keywords: underactuated system, Inertia wheel inverted pendulum, Stabilization, Sliding mode control.
Słowa kluczowe: Niedostatecznie uruchamiany system, Odwrócone wahadło koła bezwładności, Sterowanie w trybie ślizgowym.

Introduction

Mechanical systems have been used by humans for centuries to help them perform difficult or strain-intensive missions that are beyond their physical capabilities. With the evolution of robotics nowadays and with the delicate problems of modeling and controlling complex mechanical systems, the tools used are becoming more and more sophisticated [1], [2], [3].

One of the most important axes of research in the field of robotics concerns the control of mechanical systems which are divided into three types of actuated mechanisms with respect to the numbers of actuators with the number of their degrees of freedom. When there are more actuators than joints in a mechanism, it is said to be a redundant or overactuated system [4], [5].

A fully actuated system has as many actuators as it has degrees of freedom. Finally, the type of system that we study below is that of underactuated systems that have fewer actuators than degrees of freedom [6], [7].

Underactuated systems have fewer actuators than degrees of freedom. This results in the presence of generally nonlinear and non-integrable dynamic constraints. Underactuation can be found in several situations, for example in the case of vehicles such as planes, helicopters, underwater robots [6], [8]

It can also be intentionally introduced during the design of the system to reduce its weight and cost of production. Finally, when an actuator failure occurs, a fully actuated mechanical system can become underactuated and exhibit the same properties and difficulties [2], [7]

The inverted pendulum is a very interesting classic underactuated system and widely studied in the automation community, given its non-linear and unstable dynamics. It has always been an interesting challenge to control it. We find different forms of inverted pendulum, the best known are: the inverted pimple pendulum, the inverted double pendulum, the inverted pendulum of Furuta, the inverted gyroscopic pendulum [3], [7], [9].
In our case, the study focused on the Inertia wheel inverted pendulum because it is practical for studying this type of under-actuated system given its price and its ease of use.

The objective of this article is to stabilize the system by bringing it back to its point of unstable equilibrium and to maintain it in this position despite the presence of occasional external disturbances. For this, it is necessary to study and apply a robust control capable of respecting the specifications. The sliding mode control is chosen to be applied to our study system and to validate its robustness by simulations.

Sliding mode control of a underactuated system

Consider an underactuated system defined as following:

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = f\left(x_1, x_2, x_3\right) \\ &\dot{x}_3 = x_4 \\ &\dot{x}_4 = g\left(x_1, x_2, x_3\right) + b\left(x_1, x_2, x_3\right)u + d \end{aligned} \tag{1}$$

where u is control input, d is disturbance.
The function $f\left(x_1, x_2, x_3\right)$ must be satisfied to the suppositions as follows [10] :

Supposition 1: $for\ x_1 = x_2 = x_3 = 0,\ f\left(x_2, x_2, x_3\right) \to 0$;

Supposition 2: $\dfrac{\partial f}{\partial x_3}$ is invertible;

Supposition 3: $x_3 \to 0, if\ \ x_1 = x_2 = 0,\ f\left(x_1, x_2, x_3\right) \to 0$;

Supposition 4: $\left|\dfrac{\partial f}{\partial x_3}\right| \le \beta_3, i = 1,2,3.$

The control objectives for the systems are $\left(x_1, x_2, x_3, x_4\right) \to 0, when\ t \to \infty$. We define the following relations [11]:

$$\begin{cases} e_1 = x_1 \\ e_2 = \dot{e}_1 = x_2 \\ e_3 = \ddot{e}_1 = \dot{x}_2 = f\left(x_1, x_2, x_3\right) \\ e_4 = \dddot{e}_1 = \dot{f} = \dfrac{\partial f}{\partial x_1} x_2 + \dfrac{\partial f}{\partial x_2} f + \dfrac{\partial f}{\partial x_3} x_4 \end{cases} \tag{2}$$

The strategy of sliding mode controller design is based on determining the sliding surface in the first place. The design sliding mode function is defined by the following equation:

(3) $$s=\sum_{i=1}^{i=3}\left(c_i e_i\right)+e_4 \text{ , where } c_i \succ 0$$

Then

(4)
$$\begin{aligned}\dot{s} &= c_1\dot{e}_1+c_2\dot{e}_1+c_3\dot{e}_3+\dot{e}_4\\ &= c_1 x_2+c_2 f+c_3\left(\frac{\partial f}{\partial x_1}x_2+\frac{\partial f}{\partial x_2}f+\frac{\partial f}{\partial x_3}x_4\right)\\ &+\frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_2}f\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)\\ &+\frac{d}{dt}\left(\frac{\partial f}{\partial x_3}x_4\right)+\frac{\partial f}{\partial x_3}\left(g+bu+d\right)\end{aligned}$$

Considering that the derivative of the sliding surface is zero $\dot{s}=0$, so we can get equivalent control described by the following equation [10]:

(5)
$$u_{eq}=-\left(\frac{\partial f}{\partial x_3}b\right)^{-1}\begin{bmatrix} c_1 x_2+c_2 f+\\ c_3\left(\frac{\partial f}{\partial x_1}x_2+\frac{\partial f}{\partial x_2}f+\frac{\partial f}{\partial x_3}x_4\right)\\ +\frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_2}f\right)+\\ \frac{d}{dt}\left(\frac{\partial f}{\partial x_3}x_2\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_3}x_4\right)+\frac{\partial f}{\partial x_3}g\end{bmatrix}$$

To satisfy $s\dot{s}\le 0$, the sliding mode controller can be designed as [11]:

(6) $$u=u_{eq}+u_{sw}$$

where $u_{sw}=-\left(\frac{\partial f}{\partial x_3}b\right)^{-1}\left[M\,sign(s)+\lambda s\right],\lambda\succ 0$

substituting equation (6) into equation (4), we obtain :

(7)
$$\begin{aligned}\dot{s} &= c_1\dot{e}_1+c_2\dot{e}_2+c_3\dot{e}_3+\dot{e}_4\\ &= c_1 x_2+c_2 f+c_3\left(\frac{\partial f}{\partial x_1}x_2+\frac{\partial f}{\partial x_2}f+\frac{\partial f}{\partial x_3}x_4\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)+\\ &\frac{d}{dt}\left(\frac{\partial f}{\partial x_2}f\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_3}x_4\right)+\frac{\partial f}{\partial x_3}\left(g+bu+d\right)\\ &= -M\,sign(s)-\lambda s+\frac{\partial f}{\partial x_3}d\end{aligned}$$

Define

(8) $$M=\beta_3 D+\rho \text{ , } \rho\succ 0$$

Design the Lyapunov function as $V=\frac{1}{2}s^2$, then the derivative of the Lyapunov function $\dot{V}$ is defined by the following equation [12], [13]:

$$\begin{aligned}\dot{V} &= s\dot{s}=s\left[-\left(\beta_3 D+\rho\right)sign(s)-\lambda s+\frac{\partial f}{\partial x_3}d\right]\\ &= -\left(\beta_3 D+\rho\right)|s|-\lambda s^2+s\frac{\partial f}{\partial x_3}d\le-\rho|s|-\lambda s^2\le 0\end{aligned}$$

from

$$\dot{e}_1=x_2$$

$$\dot{e}_2=\dot{x}_2=f$$

$$\dot{e}_3=\dot{f}=\frac{\partial f}{\partial x_1}x_2+\frac{\partial f}{\partial x_2}f+\frac{\partial f}{\partial x_3}x_4$$

$$\begin{aligned}\dot{e}_4 &= \frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_2}f\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_3}x_4\right)\\ &= \frac{d}{dt}\left(\frac{\partial f}{\partial x_1}x_2\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_2}f\right)+\frac{d}{dt}\left(\frac{\partial f}{\partial x_3}\right)x_4\\ &+\frac{\partial f}{\partial x_3}\left(f+bu+d\right)\end{aligned}$$

From equation (2), we have $\dot{e}_1=e_2$, $\dot{e}_2=e_3$ and $\dot{e}_3=e_4$. $s\dot{s}\le 0$ indicates that there exists $s=0$ as $t\succ t_1$, when $s=0$, we have $e_4=-c_1e_1-c_2e_2-c_3e_3$ [10].

Define the matrix $A=\begin{bmatrix}0&1&0\\0&0&1\\-c_1&-c_2&-c_3\end{bmatrix}$, $E=\begin{bmatrix}e_1&e_2&e_3\end{bmatrix}$

then we have

(9) $$\dot{E}=AE$$

We consider as a choice that $Q=Q^T\succ 0$, if we design the matrix A as Hurwitz, so there exists a Lyapunov equation $A^TP+PA=-Q,\ P=P^T\succ 0$

We adopt the Lyapunov function as $V_1=E^TPE$ then,

$$\begin{aligned}\dot{V}_1 &= \dot{E}^TPE+E^TP\dot{E}=\left(AE\right)^TPE+E^TP\left(AE\right)\\ &= E^TA^TPE+E^TPAE=E^T\left(A^TP+PA\right)E\\ &= -E^TQE\le-\lambda_{\min}\left(Q\right)\left\|E\right\|_2^2\le 0\end{aligned}$$

where $\lambda_{\min}(Q)$ is the minimum eigenvalue of Q.

Simulations Results

The Inertia wheel inverted pendulum is a mechanical system consisting of a rod connected to a fixed support by a non-actuated pivot link and an inertial flywheel connected to the other end of the rod by an actuated pivot link at the center of the steering wheel. The mechanical structure of this system is shown schematically in Fig. 1.

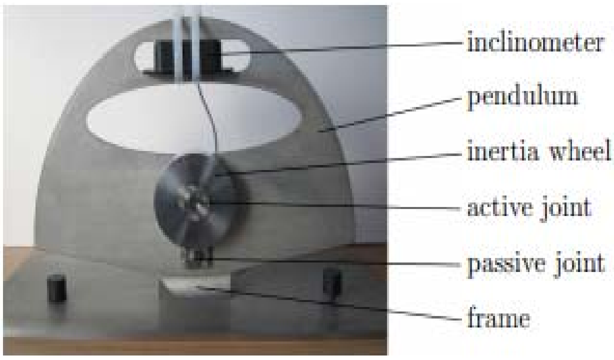

Fig.1: The inertia wheel inverted pendulum

Stabilizing the pendulum amounts to synthesizing an initial command in an unstable equilibrium position and maintaining it around this position (Fig. 2), despite the presence of external disturbances.

In addition to the stability constraints, the control must ensure the best performance while respecting the energy constraints. In addition, it is also necessary to ensure a certain robustness towards any uncertainties which are naturally present in the real model due to the use of an ideal system for the simulation or which may occur following a change in parameter such as mass.

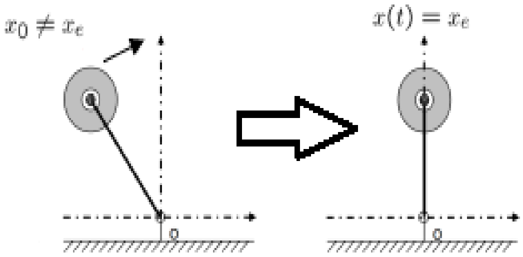

Fig.2: The inertia wheel inverted pendulum

The dynamic model of the inertia wheel inverted pendulum can be represented by the following two-order nonlinear system [14], [15], [16]:

$$\begin{cases} \ddot{\theta}_1 = \dfrac{1}{\mathrm{I}}\left[-\tau_2 + \overline{\mathrm{ml}}\,\mathrm{g}\sin\theta_1\right] \\ \ddot{\theta}_2 = \dfrac{1}{\mathrm{Ii}_2}\left[\left(\mathrm{i}_2+\mathrm{I}\right)\tau_2 - \mathrm{i}_2\overline{\mathrm{ml}}\,\mathrm{g}\sin\theta_1\right] \end{cases} \tag{10}$$

where $\mathrm{I} = \mathrm{m}_1\mathrm{l}_1^2 + \mathrm{m}_2\mathrm{l}_2^2 + \mathrm{i}_1$ $\overline{\mathrm{ml}} = \mathrm{m}_1\mathrm{l}_1 + \mathrm{m}_2\mathrm{l}_2$

Table 1 present the summary of geometric and dynamic parameters of the system.

Table 1. The parameters of the inertia wheel inverted pendulum

Description	Parameters values
Body mass	$m_1 = 3.30810\,Kg$
Wheel mass	$m_2 = 0.33081\,Kg$
Body center of mass position	$l_1 = 0.06\,m$
Wheel center of mass position	$l_2 = 0.044\,m$
Body inertia	$i_1 = 0.03146\,Kgm^2$
Wheel inertia	$i_2 = 0.00041\,Kgm^2$
Gravity acceleration	$g = 9.81\,ms^{-2}$

The state representation of the inertia wheel inverted pendulum is given by the following equation with the state vector $\mathrm{X} = \left[\mathrm{x}_1\ \mathrm{x}_2\ \mathrm{x}_3\ \mathrm{x}_4\right] = \left[\theta_1\ \dot{\theta}_1\ \theta_2\ \dot{\theta}_2\right]$.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{I}\overline{m\mathrm{l}}\,\mathrm{g}\sin x_1 - \dfrac{1}{I}\tau_2 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{\overline{m\mathrm{l}}\,\mathrm{g}\sin x_1}{I} + \dfrac{\left(i_2+I\right)}{Ii_2}\tau_2 \end{cases} \tag{11}$$

We will apply the sliding mode control to the system in order to stabilize it around its unstable equilibrium point. For this the two following scenarios were carried out:

- Nominal case
- Case of rejection external disturbances.

- ***Nominal case***

The objective of this scenario is to stabilize the inverted pendulum, the results obtained in this case are illustrated in figures 3, 4, 5 and 6. They represent the evolution of the states of the system, as well as the control input τ_2 (torque motor).

According to the evolution of the states of the system, we notice that the sliding mode controller allows to bring the inverted pendulum back to its unstable point of equilibrium and to keep it around this position

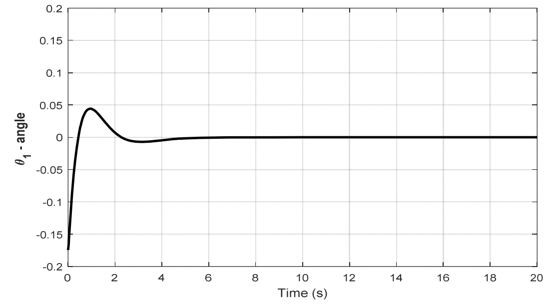

Fig.3. Angular position of the pendulum

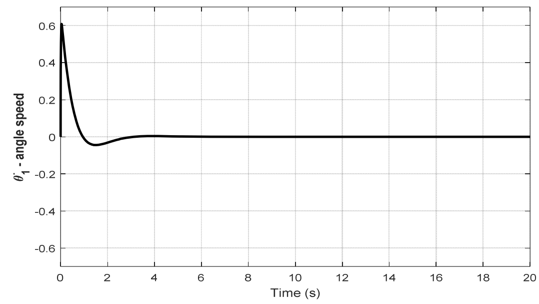

Fig.4. Angular velocity of the pendulum

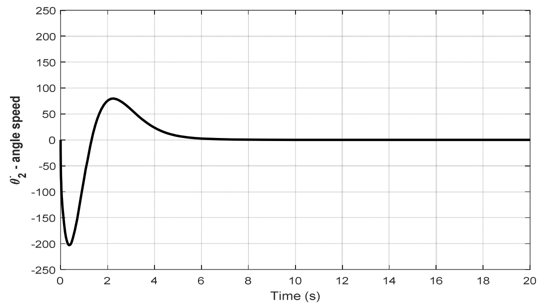

Fig.5. Inertia wheel rotation speed

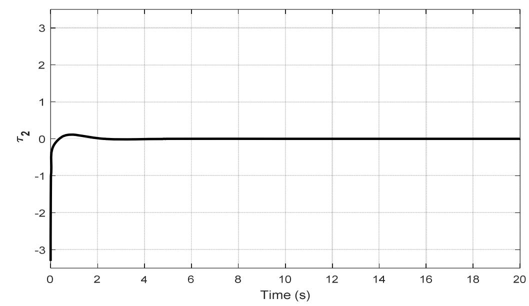

Fig.6. Control input τ_2

- ***Case of rejection external disturbances***

The objective of this scenario is to test the robustness of the proposed controller with respect to external disturbances. For this, we propose to disturb the pendulum by applying a point force to it which tends to destabilize it, in order to see the behavior of the controller and its capacity to compensate for this disturbance.

we apply a disturbance of value 0.1 N sure the inverted pendulum for a period of 0.2s (14.8s - 15s).The simulation results relating to this scenario are shown in figures 7, 8, 9 and 10. Note that the effect of these disturbances is manifested in the form of peaks on the curves. From these curves, we also notice that the sliding mode controller can reject these disturbances and bring the system back to its equilibrium position after the disturbance.

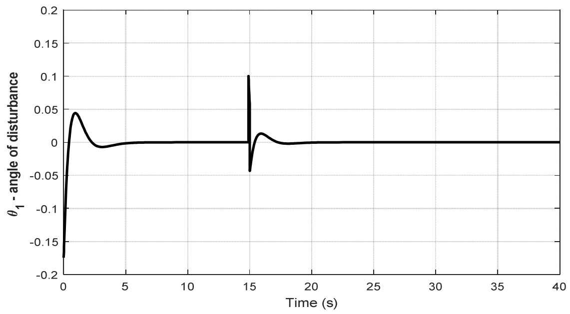

Fig.7. Angular position of the pendulum with disturbance

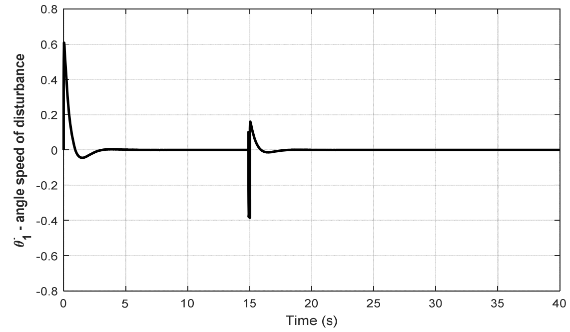

Fig.8. Angular velocity of the pendulum with disturbance

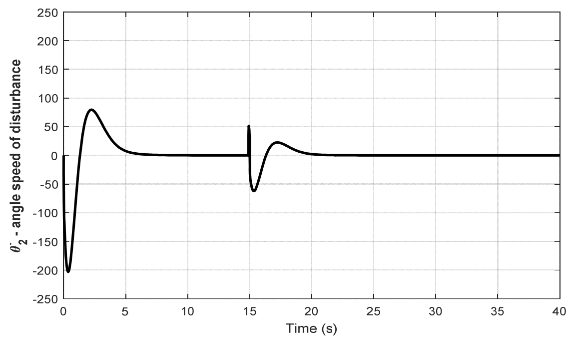

Fig.9. Inertia wheel rotation speed with disturbance

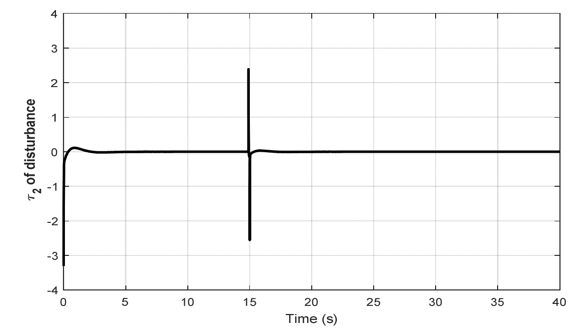

Fig.10. Control input τ_2 with disturbance

Conclusion

The work done in this article is to study an underactuated system: the inertia wheel inverted pendulum, which has two degrees of freedom and a single actuator. The Lagrangian dynamic model of the system is nonlinear and its internal dynamics are unstable. For the stabilization of the system, it is first necessary to bring the pendulum from its position of stable equilibrium (pendulum pointing downwards) to its position of unstable equilibrium (pendulum pointing upwards) and then to maintain it in this position. position despite the external disturbances that affect it. We applied a sliding mode command. For that it was necessary to adapt this command by finding a trick in order to stabilize the system. The results obtained are very interesting, they show the effectiveness of the proposed approach and that it has good performance in terms of stabilization and releases from external disturbances to the system.

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