Krzysztof TOMCZYK

Politechnika Krakowska, Katedra Automatyki i Technik Informacyjnych

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Polynomial Approximation of the Maximum Dynamic Error Generated by Measurement Systems

Abstract*. This paper presents a new approach to determining the maximum values of dynamic errors generated by a measurement system by applying polynomial approximation functions. A charge output accelerometer is used as an example of a measurement system. The maximum dynamic error were determined in terms of the response of the system to a special input signal. This signal is constrained in terms of its magnitude, which is related to the voltage sensitivity of the accelerometer. The mechanical construction of the accelerometer is presented, in conjunction with the relevant mathematical formulas, and the procedure for determining the maximum dynamic error in relation to the absolute error criterion is also discussed. Mathematical relationships for the polynomial approximation of the maximum dynamic error are presented. Based on the parameters assumed for the mathematical model of the accelerometer, the relationship between the maximum dynamic error and the period of accelerometer testing is developed. A polynomial approximation of the errors is made, and the related mathematical functions are determined for one parameter of the accelerometer. Finally, at a time corresponding to the steady state of the characteristic of maximum dynamic error, the relationship between the error and the two accelerometer parameters is derived.*

Streszczenie. *W artykule przedstawiono nowe podejście do określania maksymalnych wartości błędów dynamicznych generowanych przez układ pomiarowy, poprzez zastosowanie wielomianowych funkcji aproksymujących. Jako przykład układu pomiarowego zastosowano akcelerometr z wyjściem ładunkowym. Maksymalne błędy dynamiczne zostały określone na podstawie reakcji układu na specjalny sygnał kalibrujący. Ten sygnał jest ograniczony ze względu na jego amplitudę, która związana jest z czułością napięciową akcelerometru. Przedstawiono konstrukcję mechaniczną akcelerometru wraz z odpowiednimi formułami matematycznymi oraz omówiono procedurę określania maksymalnych błędów dynamicznych w odniesieniu do kryterium błędu bezwzględnego. Przedstawiono zależności matematyczne dla wielomianowej aproksymacji maksymalnych błędów dynamicznych. W oparciu o założone parametry matematycznego modelu akcelerometru, opracowano zależność między maksymalnymi błędami dynamicznymi i czasami badania akcelerometru. Wykonano wielomianową aproksymację błędów oraz wyznaczono matematyczne funkcje dla jednego parametru akcelerometru. Finalnie, dla czasu odpowiadającemu ustalonemu stanowi charakterystyki maksymalnego błędu, wyznaczono zależność pomiędzy błędem a dwoma parametrami akcelerometru.* ***Aproksymacja wielomianowa maksymalnego błędu dynamicznego generowanego przez systemy pomiarowe***

Keywords Polynomial approximation, charge output accelerometer, maximum dynamic error.
Słowa kluczowe: Aproksymacja wielomianowa, akcelerometr z wyjściem ładunkowym, maksymalny błąd dynamiczny.

Introduction

Determination of maximum dynamic error [1] first requires the synthesis of a mathematical model for the considered measurement system. This model must be developed in such a way that it allows the associated impulse response to be calculated. This condition is met for models based on differential equations, transfer functions, complex frequency responses or state equations. Mathematical model of most measurement systems (e.g. sensors) have a strictly defined structure and order. The parameters of this structure are determined by parametric identification [2–5] in accordance with the guidelines included in the dedicated standard [6]. This identification consists of two main stages, the first of which involves a practical experiment to determine the measurement of time or the frequency responses of the system. The second stage involves the approximation of this measurement by means of the extended least squares method, and aims to determine the values of particular model parameters [7–11]. The calculations at this stage can be conveniently performed using mathematical software (MATLAB, MathCAD, etc.), due to the necessity of carrying out advanced computations involving vector and matrix calculus. In this method, many procedures are used that are described both in international standards and in the scientific literature [10–13]. These procedures are used in routine calibration tests of measurement systems, which are carried out both in companies and in scientific centres.

When a mathematical model of the system has been developed, an appropriate mathematical procedure should be applied to determine the maximum dynamic error that can be generated by this system.

The shape of the input signal (both the number and the switching times) is also determined, as is the constraint on its magnitude [14]. In response to this signal, the maximum value of the dynamic error is obtained; any other real signal with switching times corresponding to the switching of the input signal (i.e. a change in signal sign) and within its constraint, will always generate a lower error value [1].

The abovementioned mathematical procedure can be applied to carry out comparative testing of various measurement systems, described using an analogous mathematical model with different values for its parameters. However, one substantial difficulty is the need to implement dedicated computer programs to execute these procedures [15].

In view of these difficulties, this paper proposes the application of polynomial approximations [16–18] to determine functions that describe the maximum dynamic error. These errors are produced by a measurement system with a predefined range of changeability in the selected parameters of its mathematical model.

This approximation is performed for a series of calculations of the maximum dynamic error that have been made previously. In this way, a grid of measurement points is determined that form the basis for these calculations. However, one unresolved issue is the selection of the appropriate structure and order of the polynomial function for which the lowest value of uncertainty of such an approximation is obtained. This function then provides a basis for the straightforward determination of the maximum dynamic error generated by the measurement system.

A charge output accelerometer is used in this paper as an example of measurement system, with the corresponding mathematical relations [15, 19, 20].

The structures and polynomial orders used in the approximation were determined based on the Pascal triangle, and by using the chi-square test [18].

Mathematical model of a charge output accelerometer

The mechanical construction of the charge output accelerometer is shown in Fig. 1, where $y(t)$, $u(t)$ and $x(t)$

denote relative mass displacement, absolute mass displacement and excitation (input signal), respectively.

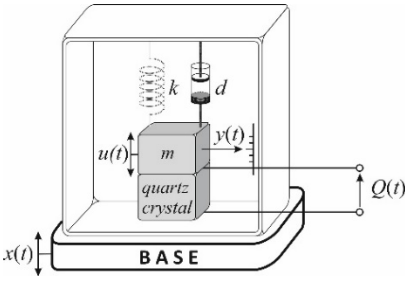


Fig. 1. Mechanical construction of the charge output accelerometer

The response of the accelerometer to a force $F(t)$ acting on the quartz crystal is

(1) $u(t) = x(t) + y(t)$

The electric charge $Q(t)$ generated by the force is represented by

(2) $Q(t) = k_p F(t)$

where $k_p = 2.2 \cdot 10^{-12} [C/N]$ denotes the piezoelectric constant [19, 20].
The transfer function of the charge output accelerometer, connected toboth voltage amplifier and the cable, is represented by

(3) $$K_{Qe}(s) = S_V \frac{\tau(2\beta\omega_0 s^2 + \omega_0^2 s)}{\tau s^3 + (2\tau\beta\omega_0 + 1)s^2 + (\tau\omega_0^2 + 2\beta\omega_0)s + \omega_0^2}$$

where $\tau = R_t C_t\ [s]$ is the time constant; R_t and C_t denote the total resistance and capacitance of the accelerometer, an associated voltage amplifier and the cable; $S_V = m/k = mS_{\mathrm{e}}\ [V/(ms^{-2})]$ is the voltage sensitivity; $m[kg]$ is the seismic mass; $d[kg/s]$ is the damping coefficient; $k[N/m]$ is the spring constant; $S_e = k_p/C_t\ [V/N]$ is the electrical sensitivity; $\beta = \frac{d}{2\sqrt{km}}$ is the damping ratio; $\omega_0 = \sqrt{k/m} = 2\pi f_0$; and f_0 denotes the non-damped natural frequency [15].

Maximum dynamic error

If the input signal is constrained to a magnitude a, then the maximum value of the absolute dynamic error can be calculated by means of the formula [1]

(4) $$EA(x_0) = a \int_0^T |k_d(t)| dt, \quad t \in (0, T)$$

where T denotes the time of excitation of the accelerometer by the signal $x_0(t)$, and

(5) $$k_d(t) = \mathcal{L}^{-1}[K_{Qe}(s) - K_s(s)]$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transformation.

The second component in the square brackets of Eq. (5) is given by

(6) $$K_s(s) = \frac{n}{s^K + r_1 s^{K-1} + r_2 s^{K-2} + \dots + r_{K-1}s + r_K} = \frac{S_V}{\prod_{i=1}^{K}\left(\frac{s}{2\pi f_c} - e^{\frac{j(2i+K-1)\pi}{2K}}\right)}$$

and represents the mathematical model of reference system for which the error is determined. Eq. (6) represents a model of a *K*-th order Butterworth filter, which can easily used as a standard; f_c denotes the filter's cut-off frequency, which is equal to the frequency bandwidth of the accelerometer; and n, $r_1, r_{2,\dots,} r_K$ are the coefficients of the numerator and denominator [18].
The input signal which maximises the error results directly from the impulse response $k_d(t)$, and can be determined by

(7) $$x_0(t) = a \cdot sign[k_d(T - t)]$$

Polynomial approximation of the dynamic error

Let the elements of the vector

(8) $$\mathbf{Z} = [z_{0,}\ z_{1,} \dots, z_{J-1}]^T$$

represent the values of time T or the values of one selected parameter of a measurement system.
The vector of errors corresponding to $\mathbf{Z}$ is

(9) $$\mathbf{E} = [E(x_0)_{0,}\ E(x_0)_{1,} \dots, E(x_0)_{J-1}]^T$$

while the polynomial of order α that approximates the error has the form

(10) $$e_j(z) = g_0 + g_1 z_j + g_2 z_j{}^2 + \dots + g_\alpha z_j{}^\alpha + \varepsilon_j,$$
j

where $g_{0,}\ g_{1,} \dots, g_{J-1}$ denote the polynomial coefficients, ε is the approximation error, and $\alpha \le J - 1$.
Let z_1 and z_2 denote any two parameters of the measurement system. Then, the matrix of errors corresponding to $\mathbf{Z}_1$ and $\mathbf{Z}_2$ has the form

(11) $$\mathbf{E} = \begin{bmatrix} E(x_0)_{0,0} & \cdots & E(x_0)_{0,M-1} \\ \vdots & \ddots & \vdots \\ E(x_0)_{N-1,0} & \cdots & E(x_0)_{N-1,M-1} \end{bmatrix}$$

Results and discussion

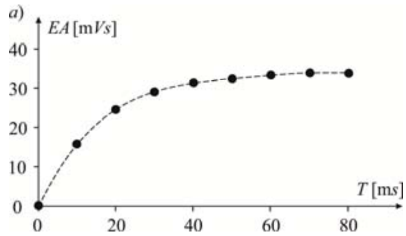
The results of calculation of the absolute error and approximation functions for the charge output accelerometer, used here as an example of measurement system, are presented and discussed below.
The values of absolute error $EA(x_0)$ in relation to time T are given in Table 1, and are obtained for the following parameters of the accelerometer: $S_{\mathrm{V}} = 1\,V/(ms^{-2})$, $\beta = 0.01$, $f_0 = 1\mathrm{k}Hz$, $\tau = 0.1\mathrm{m}s$ and the frequency bandwidth of 300 *Hz*. A tenth-order Butterworth filter was applied as a reference system, with a cut-off frequency f_c equal to the frequency corresponding to the accelerometer bandwidth. The error was calculated using Eqs. (4)−(6) for nine values of time T in the range 0−80 m*s* with the step of 10 m*s*.

Table 1. Values of the absolute error $EA(x_0)$ vs. time *T*

$T[\mathrm{ms}]$	$EA(x_0)$ [mVs]	$T[\mathrm{ms}]$	$EA(x_0)$ [mVs]
0	0	50	32.6
10	15.9	60	33.3
20	24.6	70	33.9
30	28.9	80	33.9
40	31.3		

Fig. 2(a) shows the data from Table 1 (dotted chart) and corresponding approximation function (hatched line), determined using a fourth-order polynomial. The uncertainty $u_{\mathrm{A}}[e(T)]$ for this approximation is equal to 0.178 [18]. An example of the signal $x_0(t)$ that maximises the absolute error for *T*=80 m*s* is shown in Fig. 2(b). This signal is obtained using Eq. (7), and its magnitude a is assumed to be equal to the value of the voltage sensitivity S_V of the accelerometer. A signal of any other shape will generate an error lower than that obtained for *T*=80 m*s*. As shown in Fig. 2 b), the signal $x_0(t)$ has approximately regular time switching; however, this is not a general rule, and the regularity of the switching depends on the value of the parameters β and f_0.

a)



b)

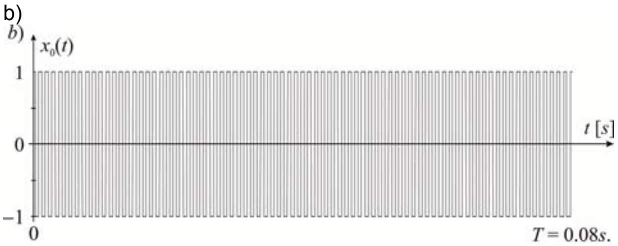


Fig. 2. *a*) Absolute error $EA(x_0)$ vs. time *T; b*) signal $x_0(t)$ obtained for *T*=80 m*s*

The approximation function determined by Eq. (10) is

$$e(T) = 0.0475 \cdot 10^{-3} + 2.02 \cdot T - 0.0504 \cdot T^2 + 0.591 \cdot \cdot 10^{-3} \cdot T^3 - 2.63 \cdot 10^{-6} \cdot T^4.$$

Based on Fig. 2(a), it can be seen that the error $EA(x_0)$ increases exponentially and then reaches a constant value at time T. Using the above function, we can determine the value of absolute error for any time T within the range 0−80 ms for an accelerometer with the parameters listed in Table 1 above. For times greater than *T*=80 m*s*, the error has a constant value corresponding to the last error value in Table 1.

The errors determined for time *T*=80 m*s* and for two chosen in advance of accelerometer parameters are taken into account in the analysis below. Table 2 shows the matrix of absolute errors $EA(x_0)$ in relation to the parameters β and τ, which fall within the ranges 0.010–0.050 and 0.100–1.10, respectively, while the parameters $S_{\mathrm{V}} = 1\,V/(ms^{-2})$ and $f_0 = 1\,\mathrm{k}Hz$ have constant values. Each time the parameters β and τ are changed, the frequency bandwidth of the accelerometer must be determined.

Table 2. Matrix of the absolute error $EA(x_0)$

$EEA(x_0)$ [mVs]		τ [ms]				
		0.10	0.350	0.600	0.850	1.10
β	0.010	33.9	57.6	61.2	62.2	62.7
	0.020	17.2	29.1	30.8	31.4	31.6
	0.030	11.6	19.4	20.6	21.0	21.1
	0.040	8.75	14.6	15.5	15.8	15.9
	0.050	7.08	11.8	12.5	12.7	12.8

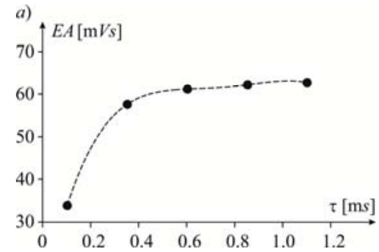
Fig. 3 shows the relationship between the absolute error and:
a) the time constant for $S_{\mathrm{V}} = 1\,V/(ms^{-2})$, $f_0 = 1\,\mathrm{k}Hz$ and $\beta = 0.010$,
b) the damping ratio for $S_{\mathrm{V}} = 1\,V/(ms^{-2})$, $f_0 = 1\,\mathrm{k}Hz$ and $\tau = 0.10$.
The corresponding approximation functions obtained for the fourth-order polynomial are:

$$e(\tau) = 11.9 + 270 \cdot \tau - 553 \cdot \tau^2 + 498 \cdot \tau^3 - 164 \cdot \tau^4,$$

$$e(\beta) = 76.8 - 6.28 \cdot 10^3 \cdot \beta + 2.38 \cdot 10^5 \cdot \beta^2 - 4.22 \cdot 10^6 \cdot \cdot \beta^3 + 2.83 \cdot 10^7 \cdot \beta^4,$$

with uncertainties $u_{\mathrm{A}}[e(\tau)] = 1.24 \cdot 10^{-8}$ and $u_{\mathrm{A}}[e(\beta)] = 6.71 \cdot 10^{-8}$, respectively.

a)



b)

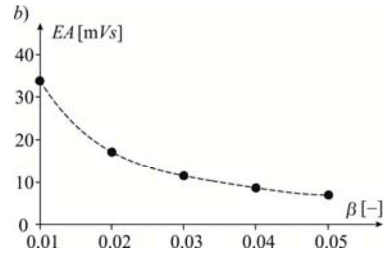


Fig. 3. *a*) Absolute error $EA(x_0)$ vs. τ for *T*=80 m*s; b*) absolute error $EA(x_0)$ vs. β for *T*=80 m*s*

Fig. 4 shows the relationship between the absolute error $EA(x_0)$, the damping ratio β and the time constant τ (dotted chart), and the three-dimensional spatial function that approximates the error. This figure shows that for the grid of 25 points from Table 2, a very close approximation was obtained, with an approximation uncertainty $u_{\mathrm{A}}[e(\beta,\tau)]$ of 0.797.

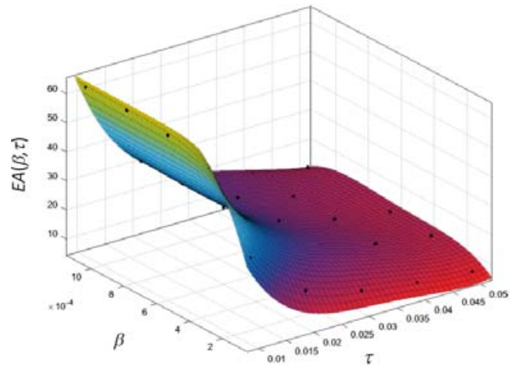


Fig. 4. Absolute error $EA(\beta,\tau)$

As a result of the approximation experiment, a fourth-order spatial approximation polynomial was obtained, as follows:

$$e(\beta,\tau) = 64.8 - 7462 \cdot \beta + 2.82 \cdot 10^8 \cdot \tau + 3.31 \cdot 10^5 \cdot \beta^2 - 8.76 \cdot 10^6 \cdot \beta \cdot \tau - 4.20 \cdot 10^8 \cdot \tau^2 - 6.40 \cdot 10^6 \cdot \beta^3 + 1.21 \cdot 10^8 \cdot \beta^2 + 6.91 \cdot 10^9 \cdot \beta \cdot \tau^2 + 2.91 \cdot 10^{11} \cdot \tau^3 + 4.51 \cdot 10^7 \cdot \beta^4 - 6.22 \cdot 10^8 \cdot \beta^3 \cdot \tau - 3.96 \cdot 10^{10} \cdot \beta^2 \cdot \tau^2 - 1.94 \cdot 10^{12} \cdot \beta \cdot \tau^3 - 7.72 \cdot 10^{13} \cdot \tau^4.$$

The structure of the above polynomial was determined using Pascal's triangle.

By substituting into the above polynomial the values of parameters β and τ, using the ranges defined by their minimum and maximum values from Table 2, we can easily obtain the value of the absolute error generated by the charge output accelerometer, for constant values of the parameters S_{V} and f_0 of $1\,V/(ms^{-2})$ and $1\,\mathrm{k}Hz$, respectively.

Conclusions

Graphical and functional relationships between the error and time, or between time and selected parameters of the charge output accelerometer, are determined in this paper. These relationships are represented by two- or three-dimensional polynomial approximations and polynomial equations, respectively. The optimal order of the polynomial was determined using the chi-square test, and was assumed to be the one for which the lowest approximation uncertainty was obtained.

Based on the polynomial equations, the values of absolute error between these points can easily be determined. These are obtained by substituting the parameter values of the accelerometer model into the above equations. However, such a substitution is only possible for the parameter values from the corresponding ranges for which the polynomial function was determined. Time-consuming error determination can be avoided by using the procedure presented in Section 3, except for the first determination of errors being points to approximation.

The solutions presented in this paper can be applied to other types of measurement systems if it is possible to determine their description using related mathematical models. However, these models should be implemented in accordance with the legal regulations applicable to these systems.

The solutions developed here can also be successfully applied in multidimensional approximation. In this case, it is necessary to significantly increase the number of calculations of the absolute error. However, after carrying out the error calculations, which are performed once, and determination of the multidimensional polynomial function, we can quickly obtain the absolute error generated by the measurement system under consideration.

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Autor: *dr inż. Krzysztof Tomczyk, Politechnika Krakowska, Katedra Automatyki i Technik Informacyjnych, ul. Warszawska 24, 31-155 Kraków*
E-mail: ktomczyk@pk.edu.pl

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