Tadeusz KACZOREK

Bialystok University of Technology

doi:10.15199/48.2015.08.30

Positivity and stability of discrete-time and continuous-time nonlinear systems

***Abstract**. The positivity and asymptotic stability of the discrete-time and continuous-time nonlinear systems are addressed. Sufficient conditions for the positivity and asymptotic stability of the nonlinear systems are established. The proposed stability tests are based on an extension of the Lyapunov method to the positive nonlinear systems. The effectiveness of the tests are demonstrated on examples.*

***Streszczenie.** Przedstawione zostaną dodatnie i stabilne asymptotycznie nieliniowe układy dyskretne i ciągłe. Podane zostaną warunki wystarczające dodatniości i stabilności asymptotycznej układów nieliniowych. Proponowane metody badania stabilności zostaną oparte na uogólnieniu metody Lyapunova. Efektywność testów zostanie zademonstrowana na przykładach numerycznych. (**Dodatniość i stabilność dyskretnych i ciągłych układów nieliniowych**).*

Keywords: positive, discrete-time, asymptotic stability, nonlinear, Lyapunov method.
Słowa kluczowe: dodatnie, dyskretene, stabilność asymptotyczna, nieliniowe, metoda Lyapunova.

Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs [8, 9] and in the papers [15-18]. Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Laypunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in [1-7]. The positive standard and descriptor systems and their stability have been analyzed in [9, 15-19]. The positive linear systems with different fractional orders have been addressed in [16, 20] and the descriptor discrete-time linear systems in [17]. Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in [10, 13] and the positivity and linearization of nonlinear discrete-time systems by state-feedbacks in [15]. The minimum energy control of positive linear systems has been addressed in [11, 12, 14]. The stability and robust stabilization of discrete-time switched systems have been analyzed in [21, 22].

In this paper the positivity and asymptotic stability of the discrete-time and continuous-time nonlinear systems will be investigated.

The paper is organized as follows. In section 2 the definitions and theorems concerning the positivity and stability of positive discrete-time and continuous-time linear systems are recalled. Necessary and sufficient conditions for the positivity of the discrete-time nonlinear systems are established in section 3. The asymptotic stability of the positive nonlinear systems is addressed in section 4, where conditions for the stability are proposed. The conditions for the positivity of continuous-time nonlinear systems are given in section 5 and for the stability of continuous-time positive nonlinear systems in section 6. Concluding remarks are given in section 7.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, Z_+ - the set of nonnegative integers, M_n - the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries), identity matrix, I_n - the $n\times n$ identity matrix.

Positive discrete-time and continuous-time linear systems and their stability

Consider the discrete-time linear system

(1a) $x_{i+1} = Ax_i + Bu_i$, $i \in Z_+ = \{0,1,...\}$

(1b) $y_i = Cx_i + Du_i$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 1. [8, 9] The discrete-time linear system (1) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 1. [8, 9] The discrete-time linear system (1) is positive if and only if

(2) $A \in \Re_+^{n\times n}$, $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$, $D \in \Re_+^{p\times m}$.

Definition 2. [8, 9] The positive discrete-time linear system (1) is called asymptotically stable if

(3) $\lim_{i\to\infty} x_i = 0$ for any $x_0 \in \Re_+^n$.

Theorem 2. The positive discrete-time linear system (1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the polynomial

(4) $p_n(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

2) All principal minors of the matrix $\bar{A} = I_n - A = [\bar{a}_{ij}]$ are positive, i.e.

(5) $M_1 = |\bar{a}_{11}| > 0,\ \ M_2 = \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{vmatrix} > 0,\ \ ..., M_n = \det \bar{A} > 0$

Proof is given in [9].

Consider the continuous-time linear system

(6a) $\dot{x} = Ax + Bu$,

(6b) $y = Cx + Du$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 3. [8, 9] The continuous-time linear system (6) is called (internally) positive if $x \in \Re_+^n$, $y \in \Re_+^p$, $t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u \in \Re_+^m$, $t \geq 0$.

Theorem 3. [8, 9] The continuous-time linear system (6) is positive if and only if

(7) $A \in M_n$, $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$, $D \in \Re_+^{p\times m}$.

Definition 4. [8, 9] The positive continuous-time linear system (6) is called asymptotically stable if

(8) $\lim\limits_{t\to\infty} x = 0$ for any $x_0 \in \Re_+^n$.

Theorem 4. The positive continuous-time linear system (6) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the polynomial

(9) $p_n(s) = \det[I_n s - A] = s^n + \hat{a}_{n-1}s^{n-1} + ... + \hat{a}_1 s + \hat{a}_0$

are positive, i.e. $\hat{a}_k > 0$ for $k = 0,1,...,n-1$.

2) All principal minors of the matrix $\hat{A} = -A = [\hat{a}_{ij}]$ are positive, i.e.

(10) $\hat{M}_1 = |\hat{a}_{11}| > 0,\ \hat{M}_2 = \begin{vmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{vmatrix} > 0,\ ...,\hat{M}_n = \det\hat{A} > 0$

Proof is given in [9].

Positivity of discrete-time nonlinear systems

Following [18] consider the discrete-time nonlinear system

(11a) $x_{i+1} = Ax_i + f(x_i,u_i)$, $i \in Z_+ = \{0,1,...\}$,

(11b) $y_i = g(x_i,u_i)$,

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$, $i \in Z_+$ are the state, input and output vectors $f(x_i,u_i) \in \Re^n$, $g(x_i,u_i) \in \Re^p$ are continuous vector functions of x_i and u_i satisfying the conditions $f(0,0) = 0$, $g(0,0) = 0$ and $A \in \Re^{n\times n}$.

Definition 5. The discrete-time nonlinear system (11) is called (internally) positive if $x_i \in \Re_+^n$, $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$.

Theorem 5. The discrete-time nonlinear system (1) is positive if and only if

(12) $A \in \Re_+^{n\times n}$ and $f(x_i,u_i) \in \Re_+^n$, $g(x_i,u_i) \in \Re_+^p$

for all $x_i \in \Re_+^n$ and $u_i \in \Re_+^m$, $i \in Z_+$.

Proof. *Sufficiency*. From (11) for $i = 0$ we have

(13) $x_1 = Ax_0 + f(x_0,u_0) \in \Re_+^n$, $y_0 = g(x_0,u_0) \in \Re_+^p$

since (12) holds and $x_0 \in \Re_+^n$, $u_0 \in \Re_+^m$.

Similarly, for $i = 1$ we obtain

(14) $x_2 = Ax_1 + f(x_1,u_1) \in \Re_+^n$, $y_1 = g(x_1,u_1) \in \Re_+^p$

since (12) and (13) holds.

Repeating the procedure for i = 2,3,… we obtain $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$ for $i \in Z_+$ and by Definition 5 the system is positive.

Necessity. Assuming that the system (11) is positive we shall show that (12) holds. From (13) for $f(x_0,u_0) = 0$ we have $x_1 = Ax_0$ and this implies $A \in \Re_+^{n\times n}$ since by assumption $x_1 \in \Re_+^n$ and $x_0 \in \Re_+^n$ can be arbitrary. If $Ax_0 = 0$ then from (13) we have $x_1 = f(x_0,u_0)$ and this implies $f(x_0,u_0) \in \Re_+^n$ since by assumption $x_1 \in \Re_+^n$. From (13) we have also $y_0 = g(x_0,u_0)$ and $x_0 \in \Re_+^n$, $u_0 \in \Re_+^m$ since by assumption $y_0 \in \Re_+^p$. Continuing the procedure we can show that (12) holds if the system is positive. □

From Theorem 5 we have the following.

Corollary 1. The discrete-time nonlinear system (3.1) is positive only if the linear system

(15) $x_{i+1} = Ax_i$, $i \in Z_+ = \{0,1,...\}$

is positive.

Example 1. Consider the discrete-time nonlinear system (11) with

(16)
$$x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix},\ A = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.4 \end{bmatrix},\ f(x_i,u_i) = \begin{bmatrix} x_{1,i}x_{2,i} + e^{-i} \\ x_{2,i}^2 + 1 - e^{-i}\cos i \end{bmatrix},$$
$$g(x_i,u_i) = \begin{bmatrix} x_{1,i}^2 + 0.1e^{-i} \\ x_{2,i} + 2 + \cos i \end{bmatrix}.$$

From (16) it follows that the matrix A has nonnegative entries and the vector functions $f(x_i,u_i)$, $g(x_i,u_i)$ are also nonnegative for all $x_i \in \Re_+^n$ and $u_i \in \Re_+^m$, $i \in Z_+$. Therefore, by Theorem 5 the system is positive. The linear part of the system is also asymptotically stable since the coefficients of the polynomial

(17) $\det[I_2(z+1) - A] = \begin{vmatrix} z+0.8 & -0.1 \\ -0.3 & z+0.6 \end{vmatrix} = z^2 + 1.4z + 0.45$

has positive coefficients, i.e. $a_0 = 0.45$, $a_1 = 1.4$.

The same result follows from the condition 2 of Theorem 2 since

(18) $\overline{A} = I_2 - A = \begin{bmatrix} 0.8 & -0.1 \\ -0.3 & 0.6 \end{bmatrix}$

and $M_1 = 0.8$, $M_2 = \det\overline{A} = 0.45$.

Stability of discrete-time positive nonlinear systems

Consider the positive discrete-time nonlinear system

(19) $x_{i+1} = Ax_i + f(x_i)$, $x_0 \in \Re_+^n$,

where $x_i \in \Re_+^n$, $A \in \Re_+^{n\times n}$, $f(x_i) \in \Re_+^n$ is a continuous and bounded vector function.

Definition 6. The positive discrete-time nonlinear system (19) is called asymptotically stable in the region $D \in \Re_+^n$ if $x_i \in \Re_+^n$, $i \in Z_+$ and

(20) $\lim\limits_{i\to\infty} x_i = 0$ for any finite $x_0 \in D \in \Re_+^n$.

To test the asymptotic stability of the positive system (19) the Lyapunov method will be used. As a candidate of Lyapunov function we choose

(21) $V(x_i) = c^T x_i > 0$ for $x_i \in \Re_+^n$

where $c \in \Re_+^n$ is a vector with strictly positive components $c_k > 0$ for $k = 1,...,n$.
Using (21) and (19) we obtain

(22) $$\begin{aligned}\Delta V(x_i) &= V(x_{i+1}) - V(x_i) = c^T x_{i+1} - c^T x_i \\ &= c^T\{[A - I_n]x_i + f(x_i)\} < 0\end{aligned}$$

for

(23) $[I_n - A]x_i - f(x_i) < 0$, $x_i \in D \in \Re_+^n$

since $c \in \Re_+^n$ is strictly positive vector.
Therefore, the following theorem has been proved.
Theorem 6. The positive discrete-time nonlinear system (19) is asymptotically stable in the region $D \in \Re_+^n$ if the condition (23) is satisfied.

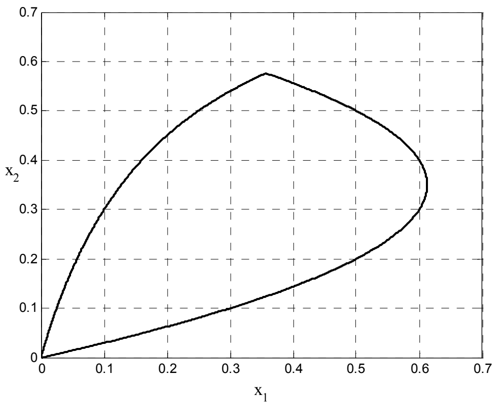

Fig. 1. Stability region (inside the curved line).

Example 2. Consider the nonlinear system (19) with

(24) $$x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \quad A = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}, \quad f(x_i) = \begin{bmatrix} x_{1,i}x_{2,i} \\ x_{2,i}^2 \end{bmatrix}.$$

The nonlinear system is positive since $A \in \Re_+^{2\times 2}$ and $f(x_i) \in \Re_+^2$ for all $x_{1,i} \geq 0$ and $x_{2,i} \geq 0$, $i \in Z_+$.
In this case the condition (23) is satisfied in the region *D* defined by

(25) $$\begin{aligned} D &:= \{x_{1,i}, x_{2,i}\} = [I_2 - A]x_i - f(x_i) \\ &= \begin{bmatrix} 0.9x_{1,i} - 0.2x_{2,i} - x_{1,i}x_{2,i} \\ 0.7x_{2,i} - 0.2x_{1,i} - x_{2,i}^2 \end{bmatrix} \in \Re_+^2 .\end{aligned}$$

The region *D* is shown on the Fig. 1.
By Theorem 6 the positive nonlinear system (19) with (24) is asymptotically stable in the region (25).

Positivity of continuous-time nonlinear system
Consider the continuous-time linear system

(26a) $\dot{x} = Ax + f(x,u)$,

(26b) $y = g(x,u)$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $f(x,u)$ and $g(x,u)$ are continuous and bounded vector functions of x and u satisfying $f(0,0) = 0$ and $g(0,0) = 0$.
Definition 7. [8, 9] The continuous-time linear system (26) is called (internally) positive if $x \in \Re_+^n$, $y \in \Re_+^p$, $t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u \in \Re_+^m$, $t \geq 0$.

Theorem 7. [8, 9] The continuous-time linear system (26) is positive if and only if

(27) $A \in M_n$, $f(x,u) \in \Re_+^n$, $g(x,u) \in \Re_+^n$
for all $x \in \Re_+^n$, $u \in \Re_+^m$, $t \geq 0$.

Proof. The solution of the equation (26a) for given A and $f(x,u)$ has the form

(28) $$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau) f[x(\tau),u(\tau)]d\tau$$

where

(29) $\Phi(t) = e^{At}$.

Using the Picard method from (28) we obtain

(30) $$x_{k+1}(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau) f[x_k(\tau),u(\tau)]d\tau, \ k = 1,2,...$$

From (29) it follows that if the conditions (27) are satisfied then $x_k(t) \in \Re_+^n$, $t \geq 0$, $k = 1,2,...$ since for $A \in M_n$, $\Phi(t) \in \Re^{n\times n}$, $t \geq 0$ [9].
From (26b) we have $y \in \Re_+^p$, $t \geq 0$ since by assumption (27) $g(x,u) \in \Re_+^n$ for $x \in \Re_+^n$, $u \in \Re_+^m$, $t \geq 0$. □

Stability of continuous-time nonlinear systems
Consider the positive continuous-time nonlinear system

(31) $\dot{x} = Ax + f(x)$,

where $x = x(t) \in \Re^n$, $A \in M_n$, $f(x) \in \Re_+^n$ is a continuous and bounded vector function and $f(0) = 0$.
Definition 8. The positive continuous-time nonlinear system (31) is called asymptotically stable in the region $D \in \Re_+^n$ if $x(t) \in \Re_+^n$, $t \geq 0$ and

(32) $\lim_{t\to\infty} x(t) = 0$ for any finite $x_0 \in D \in \Re_+^n$.

To test the asymptotic stability of the positive system (31) the Lyapunov method will be used. As a candidate of Lyapunov function we choose

(33) $V(x) = c^T x > 0$ for $x = x(t) \in \Re_+^n$, $t \geq 0$

where $c \in \Re_+^n$ is a vector with strictly positive components $c_k > 0$ for $k = 1,...,n$.
Using (33) and (31) we obtain

(34) $\dot{V}(x) = c^T \dot{x} = c^T[Ax + f(x)] < 0$

for

(35) $Ax + f(x) < 0$ for $x \in D \in \Re_+^n$, $t \geq 0$

since $c \in \Re_+^n$ is strictly positive vector.
Therefore, the following theorem has been proved.
Theorem 8. The positive continuous-time nonlinear system (6.1) is asymptotically stable in the region $D \in \Re_+^n$ if the condition (6.5) is satisfied.

Example 3. Consider the nonlinear system (31) with

(36) $$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \quad f(x) = \begin{bmatrix} x_1x_2 \\ x_2^2 \end{bmatrix}.$$

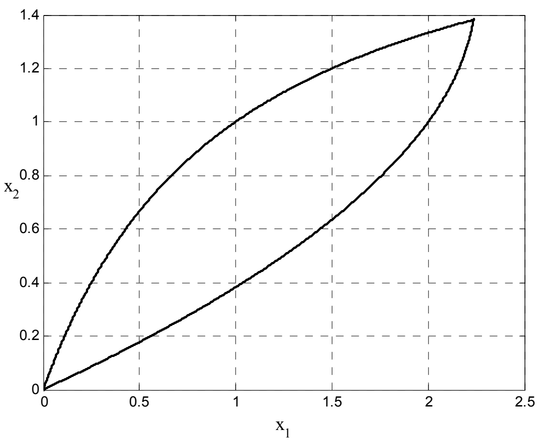

Fig. 2. Stability region (inside the curved line).

The nonlinear system (31) with (36) is positive since $A \in M_2$ and $f(x) \in \Re_+^2$ for all $x \in \Re_+^2$, $t \geq 0$.

In this case the condition (34) is satisfied in the region *D* defined by

$$(37)\quad D := \{x_1, x_2\} = \begin{bmatrix} -2x_1 + x_2 + x_1 x_2 \\ x_1 - 3x_2 + x_2^2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From (37) we have

$$(38)\quad x_1(2 - x_2) > x_2 > 0 \text{ and } 0 \leq x_1 < (3 - x_2)x_2 .$$

The region *D* is shown on the Fig. 2.

By Theorem 8 the positive nonlinear system (31) with (36) is asymptotically stable in the region (37).

Concluding remarks

The positivity and asymptotic stability of the discrete-time and continuous-time nonlinear systems have been addressed. Necessary and sufficient conditions for the positivity of the discrete-time nonlinear systems have been established (Theorem 5). Using the Lyapunov direct method the sufficient conditions for asymptotic stability of the discrete-time nonlinear systems have been proposed (Theorem 6). The effectiveness of the conditions has been demonstrated on Example 1. Sufficient conditions for the positivity of continuous-time nonlinear systems have been established in section 5 (Theorem 7) and for the asymptotic stability in section 6 (Theorem 8). The stability conditions for continuous-time nonlinear systems are illustrated by Example 3. The considerations can be extended to fractional discrete-time nonlinear systems. An open problem is an extension of the conditions to the descriptor fractional discrete-time and continuous-time nonlinear systems.

Acknowledgment

This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

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***Authors**: prof. dr hab. inż. Tadeusz Kaczorek, Politechnika Białostocka, Wydział Elektryczny, ul. Wiejska 45D, 15-351 Białystok, E-mail: kaczorek@isep.pw.edu.pl.*