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Modelling of Linear Analogue Transducers in Frequency Domain

***Abstract**. The paper presents the method for modelling of linear analogue transducers based on the simultaneous measurement of the amplitude and the phase characteristics in LabVIEW program. The solutions presented are based on the transfer function reparameterisation, which is the basis for the implementation of the weighted least squares procedure [1-3]. The effectiveness of the presented method is verified using an example of the acceleration sensor PCB 338b35 modelling.*

Streszczenie.** Artykuł przedstawia metodę modelowania liniowych analogowych przetworników w oparciu o równoczesny pomiar charakterystyk amplitudowej i fazowej w programie LabVIEW. Przedstawione rozwiązania oparte są na reparametryzacji funkcji przejścia, stanowiącej podstawę do implementacji ważonej metody najmniejszych kwadratów. Efektywność przedstawionej metody została zweryfikowana na przykładzie modelowania czujnika przyspieszenia PCB 338b35.* ***Modelowania liniowych analogowych przetworników w oparciu o równoczesny pomiar charakterystyk amplitudowej i fazowej w programie LabVIEW

Keywords Analogue transducer, weighted least squares procedure, reparameterisation of transfer function.
Słowa kluczowe: Przetwornik analogowy, ważona metoda najmniejszych kwadratów, reparametryzacja funkcji przejścia.

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Introduction

In measurement practice, the modelling of the transfer function of the measuring transducers, usually defined as an identification process, is most often carried out based on the measurement of the amplitude-frequency characteristic [4-6]. However, the best mapping accuracy of a transducer can be obtained only based on the simultaneous measurement of both the amplitude and phase frequency characteristics. Such an approach to the modelling, requires the implementation of appropriate numerical algorithms, enabling accurate estimation of the parameters as well as the determination of their uncertainty.

This paper presents an application of the weighted least squares procedure to the modelling of the wide class of the measuring transducers on the basis of the measurement of both frequency characteristics. This procedure is based on the reparameterisation of the transfer functions for the most applied the measuring transducers and is examined in detail in this paper. The procedure for estimation of the uncertainties of the model parameters by means of the Monte Carlo method and by application of χ^2 test also is presented.

As an example of the application of the presented methods the modelling of the acceleration sensor *PCB 338b35* is examined in the last section of the paper. The results of the calculation have been obtained by using the MathCad program.

Application of the weighted least squares procedure

Let us consider a typical transfer function of *k*-th order as follows

(1) $$K(\omega)=\frac{a_0}{1+a_1 j\omega-a_2\omega^2+\ldots+a_k(j\omega)^k}$$

which can also be written as

(2) $$K(\omega)=A(\omega)exp[j\Phi(\omega)]= =\frac{1}{\lambda_0+\lambda_1 j\omega-\lambda_2\omega^2+\ldots+\lambda_k(j\omega)^k}=\frac{1}{\mathbf{\Theta}^T(\omega)\boldsymbol{\lambda}}$$

where

(3) $$\mathbf{\Theta}^T(\omega)=[1, j\omega, -\omega^2, \ldots, (j\omega)^k]$$

(4) $$\boldsymbol{\lambda}^T=(\lambda_0,\lambda_1,\lambda_2,\ldots,\lambda_k)=\left[\frac{1}{a_0},\frac{a_1}{a_0},\frac{a_2}{a_0},\ldots,\frac{a_k}{a_0}\right]$$

are the vectors of dimension $k+1$.

Let $\mathbf{\Xi}$ be the matrix of dimension $2N x(k+1)$

(5) $$\mathbf{\Xi}^T{}_{(2N,k+1)}=\{Re[\Theta^T(\omega_0)]\quad Re[\Theta^T(\omega_1)]\quad\cdots\quad Re[\Theta^T(\omega_n)]\quad\cdots\quad Re[\Theta^T(\omega_{N-1})]\; Im[\Theta^T(\omega_0)]\quad Im[\Theta^T(\omega_1)]\quad\cdots\quad Im[\Theta^T(\omega_n)]\cdots\cdots Im[\Theta^T(\omega_{N-1})]\}$$

where $n=0,1,\ldots,N-1$, N denotes the number of measurement points.

The procedure for determining the parameters of model (1) and their uncertainties, based on measurement of the amplitude $\mathbf{A}(\omega_n)$ and the phase $\mathbf{\Phi}(\omega_n)$ characteristics, is carried out in the following stages:

1. Determination of the vector

(6) $$\mathbf{V}^T=\{Re[A(\omega_0)^{-1}exp(-j\mathbf{\Phi}(\omega_0))]\quad Re[A(\omega_1)^{-1}exp(-j\mathbf{\Phi}(\omega_1))]\cdots\quad Re[A(\omega_n)^{-1}exp(-j\mathbf{\Phi}(\omega_n))]\cdots\quad Re[A(\omega_{N-1})^{-1}exp(-j\mathbf{\Phi}(\omega_{N-1}))]\cdots\quad Im[A(\omega_0)^{-1}exp(-j\mathbf{\Phi}(\omega_0))]\quad Im[A(\omega_1)^{-1}exp(-j\mathbf{\Phi}(\omega_1))]\cdots\quad Im[A(\omega_n)^{-1}exp(-j\mathbf{\Phi}(\omega_n))]\cdots\quad Im[A(\omega_{N-1})^{-1}exp(-j\mathbf{\Phi}(\omega_{N-1}))]\}$$

based on Eq. (2).

2. Determination of the vector

(7) $$\hat{\boldsymbol{\lambda}}=(\mathbf{\Xi}^T\mathbf{C}^{-1}\mathbf{\Xi})^{-1}\mathbf{\Xi}^T\mathbf{C}^{-1}\mathbf{V}$$

or

(8) $$\hat{\boldsymbol{\lambda}}=arg\,min\{(\mathbf{V}-\mathbf{\Xi}\boldsymbol{\zeta})^T\mathbf{C}^{-1}(\mathbf{V}-\mathbf{\Xi}\boldsymbol{\zeta})\}$$

where

(9) $$\boldsymbol{\zeta}=(\mathbf{\Xi}^T\mathbf{\Xi})^{-1}\mathbf{\Xi}^T\mathbf{V}$$

and $\mathbf{C}$ is the symmetrical and positive definite covariance matrix of the vector $\mathbf{V}$. This matrix is determined by applying the Monte Carlo method [7] as follows:
- calculation of expected values of the characteristics

(10) $E_A = E[A(\omega)],\ \ E_\Phi = E[\Phi(\omega)]$

and their standard deviations

(11) $\sigma_A = \sqrt{V[A(\omega)]},\ \ \sigma_\Phi = \sqrt{V[\Phi(\omega)]}$

- choice of the pseudo-random generator, which draw the number from the range $MC \in (10^4 - 10^6)$.
- determination of the matrices

(12) $\boldsymbol{\varepsilon}_{A(n)} = random\,[N(E_A, \sigma_A)]$

$\boldsymbol{\varepsilon}_{\Phi(n)} = random\,[N(E_\Phi, \sigma_\Phi)]$

(13) $\boldsymbol{\rho}_{A(n,m)} = A(\omega_n) + \dfrac{A(\omega_n)\boldsymbol{\varepsilon}_{A(n)}}{100}$

$\boldsymbol{\rho}_{\Phi(n,m)} = \Phi(\omega_n) + \dfrac{\Phi(\omega_n)\boldsymbol{\varepsilon}_{\Phi(n)}}{100}$

where $m = 1,2,\ldots,MC$ and $N(E,\sigma)$ is the Gaussian normal distribution of the measurement points.
- calculation of the matrix (14)

(14)
$$\boldsymbol{\Psi}_{(2N,MC)} = \begin{bmatrix} \mathrm{Re}[\boldsymbol{\rho}_{A(1,1)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(1,1)})] & \cdots & \mathrm{Re}[\boldsymbol{\rho}_{A(1,MC)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(1,MC)})] \\ \vdots & \ddots & \vdots \\ \mathrm{Re}[\boldsymbol{\rho}_{A(N,1)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(N,1)})] & \cdots & \mathrm{Re}[\boldsymbol{\rho}_{A(N,MC)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(N,MC)})] \\ \mathrm{Im}[\boldsymbol{\rho}_{A(1,1)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(1,1)})] & \cdots & \mathrm{Im}[\boldsymbol{\rho}_{A(1,MC)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(1,MC)})] \\ \vdots & \ddots & \vdots \\ \mathrm{Im}[\boldsymbol{\rho}_{A(N,1)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(N,1)})] & \cdots & \mathrm{Im}[\boldsymbol{\rho}_{A(N,MC)}{}^{-1}\exp(-j\boldsymbol{\rho}_{\Phi(N,MC)})] \end{bmatrix}$$

The covariance matrix is determined based on matrix $\boldsymbol{\Psi}$ - Eq. (14).

3. Calculation of the parameters of model (1) based on Eq. (7) or Eq. (8)

(15) $a_{0e} = \dfrac{1}{\hat{\lambda}_0},\ a_{1e} = \dfrac{\hat{\lambda}_1}{\hat{\lambda}_0},\ \cdots,\ a_{ke} = \dfrac{\hat{\lambda}_k}{\hat{\lambda}_0}$

4. Estimation of the uncertainties of the model parameters by means of the Monte Carlo method:

- calculation of the vector

(16) $\hat{\boldsymbol{\lambda}}^m = \hat{\boldsymbol{\lambda}} + \varepsilon^m$

where ε^m is a multivariate normal distribution [1] with covariance matrix $\mathbf{C}_u$ calculated by

(17) $\mathbf{C}_u = (\boldsymbol{\Xi}^T\mathbf{C}^{-1}\boldsymbol{\Xi})^{-1}$

- determination of the uncertainties $\tilde{a}_0^m, \tilde{a}_1^m, \tilde{a}_2^m, \ldots, \tilde{a}_k^m$ based on $\hat{\boldsymbol{\lambda}}^m$

The total uncertainties $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_k$ of the parameters $a_0, a_1, a_2, \ldots, a_k$ are determined as the mean value for each m.

5. Validation of the obtained model. In the case of the weighted least squares method application, the α - th quantile χ^2 test is most commonly used with

(18) $v = 2N - k$

degrees of freedom and α equals 0.05.

The number of measuring points *N* is selected so as to satisfy the condition (18).
For determining the model validity obtained by Eqs. (5)-(17), it is proposed to check the criterion [1, 8]

(19) $\chi^2_{v,1-\alpha/2} \le min\{(\boldsymbol{\Psi} - \boldsymbol{\Xi}\boldsymbol{\zeta})^T\mathbf{C}^{-1}(\boldsymbol{\Psi} - \boldsymbol{\Xi}\boldsymbol{\zeta})\} \le \chi^2_{v,\alpha/2}$

which requires an even distribution of measuring points on the frequency axis.

If condition (19) is not met, it is recommended to increase the number of measurement points.

Reparameterisation of measuring transducers transfer function

Based on Eqs. (20)–(35), the exemplary reparameterisation of the first and second order mathematical models is presented below, which are given in the form of the transfer function.
For the first order model

(20) $K(s) = \dfrac{a}{1 + s\tau}$

where a is the amplification coefficient and τ is the time constant, we have

(21) $\boldsymbol{\lambda}^T = (\lambda_0, \lambda_1) = \left[\dfrac{1}{a}, \dfrac{\tau}{a}\right]$

and

(22) $\Theta^T(\omega) = (1, j\omega)$

Estimated parameters of the model (19), based on Eq. (20) equal

(23) $a_e = \dfrac{1}{\hat{\lambda}_0},\ \ \tau_e = \dfrac{\hat{\lambda}_1}{\hat{\lambda}_0}$

For the low-pass second order model, e.g., accelerometers with seismic mass

(24) $K(s) = \dfrac{a}{s^2 + 2\beta\omega_0 s + \omega_0{}^2}$

we have

(25) $\boldsymbol{\lambda}^T = (\lambda_0, \lambda_1, \lambda_2) = \left[\dfrac{\omega_0{}^2}{a}, \dfrac{\beta\omega_0}{a}, \dfrac{1}{a}\right]$

and

(26) $\Theta^T(\omega) = (1, j2\omega, -\omega^2)$

where ω_0 is the undamped natural frequency β is the damping factor.
Estimated parameters of Eq. (24) equal

(27) $a_e = \dfrac{1}{\hat{\lambda}_2},\ \ \omega_{0e} = \sqrt{\dfrac{\hat{\lambda}_0}{\hat{\lambda}_2}},\ \ \beta_e = \dfrac{\hat{\lambda}_1}{\sqrt{\hat{\lambda}_0\hat{\lambda}_2}}$

We have a similar solution for the piezoelectric accelerometers model

(28) $$K(s) = \frac{S\omega_0^2}{s^2 + 2\beta\omega_0 s + \omega_0^2}$$

(29) $$\boldsymbol{\lambda}^T = (\lambda_0, \lambda_1, \lambda_2) = \left[\frac{1}{S}, \frac{\beta}{S\omega_0}, \frac{1}{S\omega_0^2}\right]$$

and

(30) $$\Theta^T(\omega) = (1, j2\omega, -\omega^2)$$

where S is the charge sensitivity $[\mathrm{pC/(m/s^{-2})}]$.

Estimated parameters equal

(31) $$S_e = \frac{1}{\hat{\lambda}_0},\ \omega_{0e} = \sqrt{\frac{\hat{\lambda}_0}{\hat{\lambda}_2}},\ \beta_e = \frac{\hat{\lambda}_1}{\sqrt{\hat{\lambda}_0 \hat{\lambda}_2}}$$

For the exemplary high-pass second order model, e.g. vibrometer

(32) $$K(s) = \frac{-a s^2}{s^2 + 2\beta\omega_0 s + \omega_0^2}$$

we have

(33) $$\boldsymbol{\lambda}^T = (\lambda_0, \lambda_1, \lambda_2) = \left[\frac{\omega_0^2}{a}, \frac{\beta\omega_0}{a}, \frac{1}{a}\right]$$

(34) $$\Theta^T(\omega) = \left(\frac{1}{\omega^2}, \frac{j2}{\omega}, -1\right)$$

(35) $$a_e = \frac{1}{\hat{\lambda}_2},\ \omega_{0e} = \sqrt{\frac{\hat{\lambda}_0}{\hat{\lambda}_2}},\ \beta_e = \frac{\hat{\lambda}_1}{\sqrt{\hat{\lambda}_0 \hat{\lambda}_2}}$$

Modelling of acceleration transducer PCB 338b35

An application of the algorithm discussed in the first of two sections to the modelling of the accelerometer with the voltage output is presented below. The transfer function of this accelerometer was modelled by means of Eq. (23).
The frequency characteristics were determined by means of the computer aided measuring system presented in Fig. 1. This system includes: modelling and reference accelerometer, vibration exciter, computer equipped with the NI PCI 5112 DAQ card and the LabVIEW program as well as power, measuring and charge amplifiers.

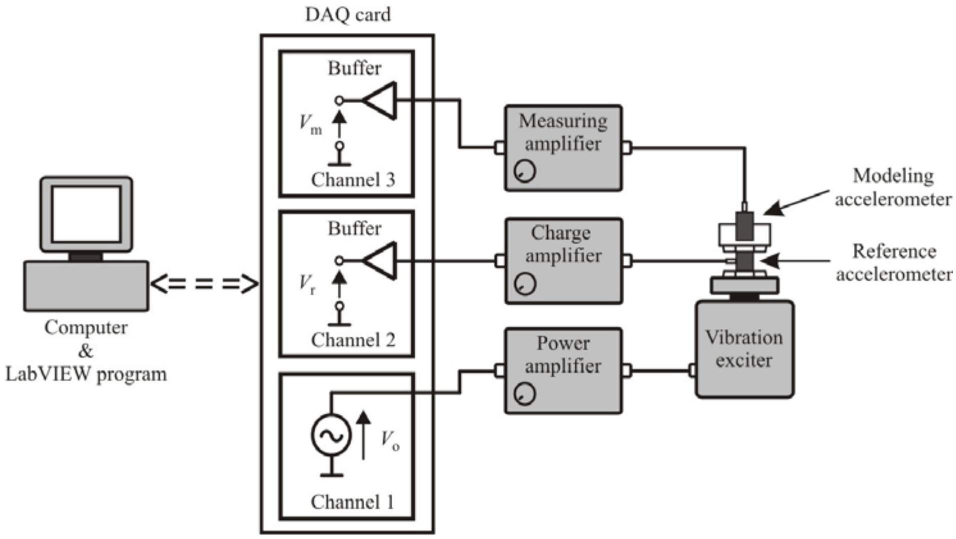

Fig. 1. Measuring system for frequency characteristics determination.

The modeled accelerometer is the type of PCB 338b35 with the sensitivity equals 10.2 $\mathrm{mV/(ms^{-2})}$. The frequency ranges for this accelerometer equal 1Hz-2kHz and 0.7Hz-3kHz for the sensitivity deviation equals 5% and 10% respectively.

The Brüel&Kjær 8305 accelerometer with a frequency ranges of 0.2Hz to 3.1kHz and 0.2Hz to 4.4kHz for the sensitivity deviation equals 1% and 2% respectively was applied as a reference. This is a charge output type accelerometer with the sensitivity equals 0.12 $\mathrm{pC/(m/s^{-2}}$*)*.

The both accelerometers which outputs V_m and V_r are connected to the measuring and charge amplifiers were excited by means of Brüel&Kjær vibration exciter type 4809 driven by Brüel&Kjær power amplifier type 2706 with the sinusoidal input signal V_o. This signal is the output of the DAQ card. The vibration exciter with a frequency range of 10Hz to 20kHz was a sinusoidal excitated with an acceleration amplitude of approximately 10 $\mathrm{ms^{-2}}$. This choice of acceleration amplitude enabled the performance of frequency characteristics measurements in the range of 20Hz to 20kHz for assumed 34 different frequencies by means of the DAQ card and the LabVIEW program. Outputs acceleration signals V_m and V_r were sampled with a rate of 2.5 10^5 samples/s.

Fig. 2 presents the block diagram of the program for simultaneous acquisition of both frequency characteristics.

The input parameters for the sinus function generator are: amplitude, frequency and sampling rate.

It was assumed that for the Monte Carlo method, the value of *MC* equals 10^5. The parameters of accelerometer (23) and their uncertainties were determined based on Eqs. (15)-(16).

Based on the measuring points of both frequency characteristics the following values of (10) and (11) were obtained $E_A = 0.98,\ E_\Phi = -0.52,\ \sigma_A = 0.09,\ \sigma_\Phi = 0.55.$

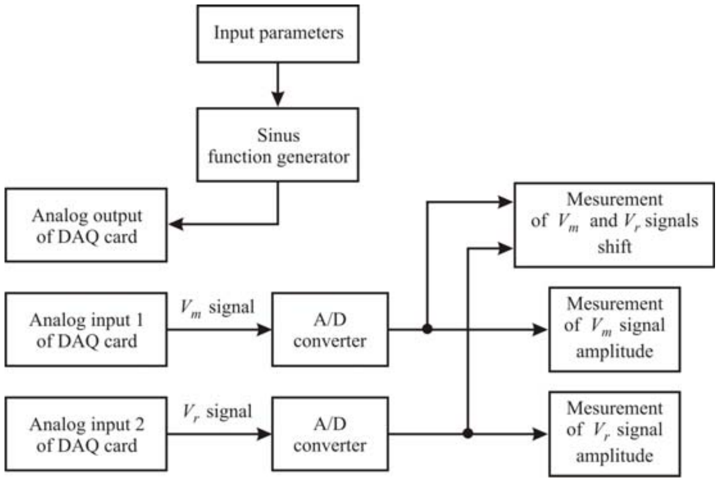

Fig. 2. Diagram of measuring system intended for determination of both frequency characteristics

Fig. 3 presents the diagram of measuring system intended for determination of both frequency characteristics in LabVIEW.

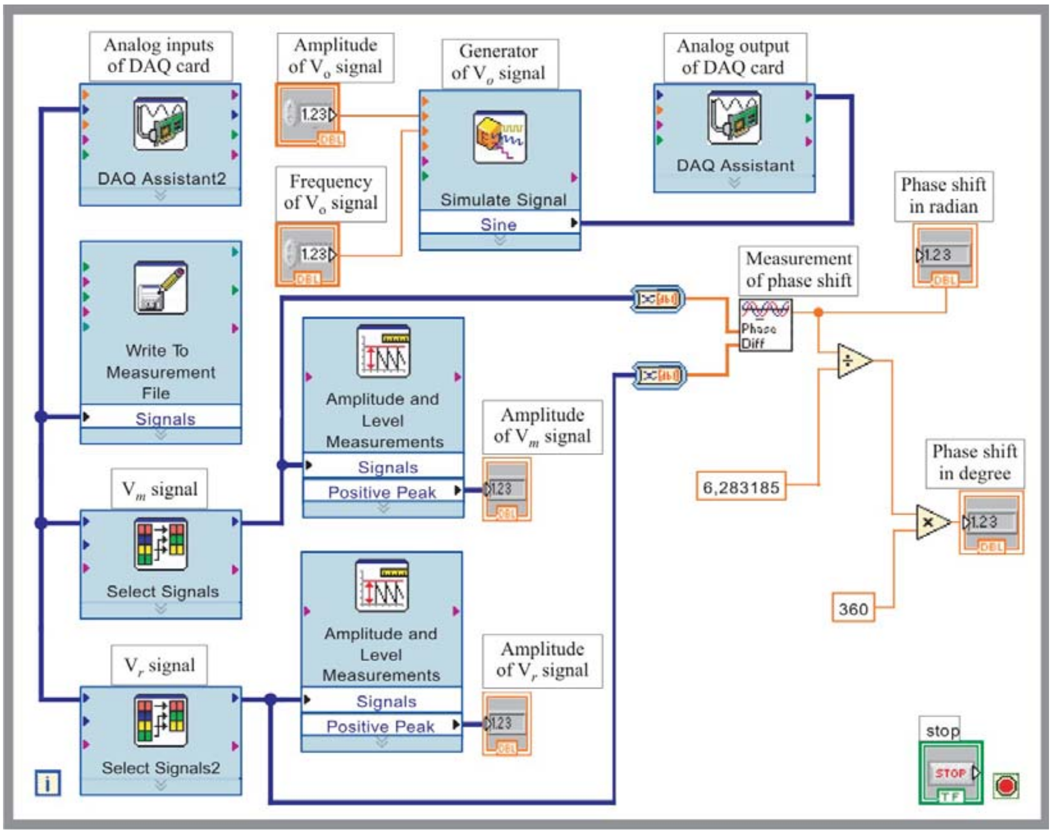

Fig. 3. Diagram of measuring system intended for determination of both frequency characteristics.

The values of the estimated parameters of model (23) with associated uncertainties are listed in Table 1.

Table 1. Estimated parameters of model (1.25) and their uncertainties.

a_e [V/V]	$3.88e+8$
f_{0e} [Hz]	$1.93e+4$
β_e	0.7
Δa_e [V/V]	$1.1e+6$
Δf_{0e} [Hz]	24.68
$\Delta \beta_e$	$1.25e-3$

The validity of the obtained model was checked by the application of the χ^2 test with 65 degrees of freedom and with $\alpha = 0.05$, as according to (18).

Taking into account that the value of $\min\left\{(\mathbf{\Psi}-\mathbf{\Xi}\zeta)^T \mathbf{C}^{-1}(\mathbf{\Psi}-\mathbf{\Xi}\zeta)\right\}$=52.08, as well as the resulting from chi-square distribution: $\chi^2_{65,\ 0.975} = 44.60$ and $\chi^2_{65,\ 0.025} = 89.18$, it means that this model passes the validity test. For 36 and 54 measuring points, the χ^2 test was failed.

Fig 3. presents frequency characteristics determined in the MathCad program according to the parameters listed in Table 1, where A and Φ denote vectors of both characteristics measuring points as well as $K(f)$ was determined based on (2). The obtained values of (2) parameters equal $\lambda_0 = 0.97$, $\lambda_1 = 3.49e-5$, $\lambda_2 = 2.57e-9$.

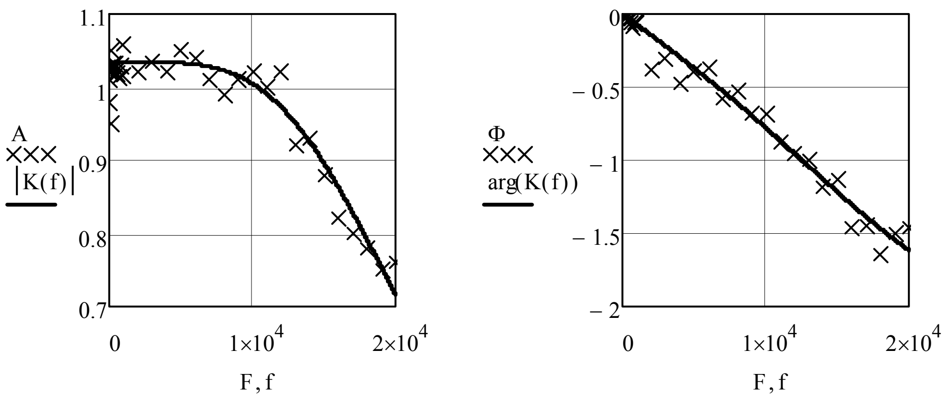

Fig. 4 Determined frequency characteristics in MathCad program.

Conclusion

The possibility of the application of the methods presented in this paper is limited to the models, the numerator of which is monomial. Otherwise, it is necessary to reduce the order of the numerator, e.g. by applying the optimisation methods presented in [9-10]. In the case of multi-inertial models, it is possible to make their transformation to the model described by Eq. (1) applying methods presented in [11].

The result of the class of transducer model adopted erroneously or mismatched number of measurement points is failure the condition (19). These incorrect assumptions are equivalent to the need of the results rejection and repetition the modelling process. Such an approach is a fundamental advantage of linear analog transducers modelling presented in the paper over the commonly used alternative methods.

In the case of noise effect, it is convenient to apply of methods presented in [12] for measuring the characteristics of *y*(*x*) defined by parametric variables *x*(*t*), *y*(*t*) or algorithms discussed in [13], which enable nonparametric or parametric frequency domain identification in the presence of nonlinear distortions under some general conditions for random multisine excitations.

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