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Cross-correlation Function Determination by Using Deterministic and Randomized Quantization

***Abstract**. The influence of quantization on the cross-correlation function determination is discussed. Three types of quantization: deterministic, dither randomized, and randomized by inputting signals into a quantizer are considered. In each case, a relation for cross-correlation function bias is given.*

***Streszczenie.** Celem artykułu jest analiza wpływu różnych rodzajów kwantowania na dokładność wyznaczania funkcji korelacji wzajemnej sygnałów. Rozważono trzy sposoby kwantowania: kwantowanie deterministyczne oraz randomizowane za pomocą sygnałów ditherowych i sygnałów wprowadzonych do kwantyzatorów. W każdym przypadku sformułowano zależności na obciążenie estymatorów funkcji.(**Wyznaczanie funkcji korelacji wzajemnej z zastosowaniem kwantowania deterministycznego i randomizowanego**).*

Keywords: cross-correlation function, estimator, deterministic quantization, randomized quantization.
Słowa kluczowe: funkcja korelacji wzajemnej, estymator, kwantowanie deterministyczne, kwantowanie randomizowane.

Introduction

The development of the measurement technique is accompanied by developing the existing as well as by devising new ways of signal quantization. The literature is constantly providing new knowledge on deterministic quantization, quantization with dither, and quantizer randomization [1-6]. It may have seemed that the huge technological progress leading, among other things, to constructing multi-bit converters would push several-level processing correlators from the market. It has not been the case, though and they are still being made, mainly for specialized measurements, e.g. in radioastronomy or in research on ionosphere dispersion. Research on correlation function estimators continues to be done to increase the equipment operational speed (of real-time measurements) or to simplify measurement procedures (of data reduction) [7]. Randomized quantization combined with averaging is aimed to improve conversion accuracy and dedicated to both low-bit and multi-bit converters.

Cross-correlation function

A cross-correlation function $R_{xy}(\tau)$ is a similarity measure of two signals $x(t)$ and $y(t)$, and for an ergodic random process, it is expressed by the relation [8]:

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\, y(t+\tau)\, \mathrm{d}t \qquad (1)$$

In the measurement technique, estimators of this function are used. In order to obtain a digital estimator, the signal undergoes discretization in the time and value domains. For sampled and quantized signals $x_q(i\Delta t)$ and $y_q(i\Delta t)$, the estimator can assume the form:

$$\tilde{R}_{xy}(k,M) = \frac{1}{M} \sum_{i=0}^{M-1} x_q(i\Delta t)\, y_q(i\Delta t + k\Delta t) \qquad (2)$$

where: Δt - sampling period, M - number of samples taken, $k\Delta t$ - delay.

Correlation function estimators can be determined by making use of the various ways of quantization mentioned in the Introduction.

Evaluation of estimator accuracy

Estimator accuracy is determined by means of the variance describing the error random component and the bias describing the systematic component [8, 9]:

$$Var\left[\tilde{R}_{xy}(k,M)\right] = E\left[\left(\tilde{R}_{xy}(k,M)\right)^2\right] - E^2\left[\tilde{R}_{xy}(k,M)\right] \qquad (3)$$

$$b\left[\tilde{R}_{xy}(k,M)\right] = E\left[\tilde{R}_{xy}(k,M)\right] - R_{xy}(k) \qquad (4)$$

Based on (3) and (4), the value of the mean square error is determined. The minimization of such error often reflects a trade-off between the minimization of the variance and the estimator bias.

Bias of the estimator obtained based on deterministically quantized signals

Quantization is an operation converting a sampled signal into a discrete signal. In most cases, the quantization intervals have an equal width q called a quantization step. When a quantity x has been quantized, the result is a quantity x_q, most frequently determined from the formula:

$$x_q = q \cdot \mathrm{ent}\left(\frac{x}{q} + \frac{1}{2}\right) \qquad (5)$$

where $\mathrm{ent}(\cdot)$ is an operator determining the integral part.

This way of quantization is described by Widrow's quantization theory. Widrow is the author of theorems concerning the conditions under which the statistics of original quantities can be reconstructed from the statistics of quantized quantities. Among other things, random variable moments are such statistics. For any set value t, the cross-correlation function of signals $x(t)$ and $y(t+\tau)$ can be considered in terms of joint moments of random variables x and y which assume the values of these signals.

Let us suppose that x_q and y_q are random variables assuming the values of quantized signals. The theorems concerning the recovery of the joint moments of the random variables x and y from the joint moments of the random variables x_q and y_q have the form [6][1]:

Quantizing Theorem II for Two Variables

The joint moments of random variables x and y can be reconstructed from the joint moments of random variables x_q and y_q if:

[1] Versions of these theorems from various periods differ somewhat.

$$\Phi_{xy}(v_1, v_2) = 0 \text{ for } |v_1| > \frac{2\pi}{q_1} \text{ and } |v_2| > \frac{2\pi}{q_2} \quad (6)$$

where $\Phi_{xy}(v_1, v_2)$ is the joint characteristic function of the variables x and y, while q_1 and q_2 are quantization intervals.

Condition (6) formulated by Widrow is not fulfilled by most signals occurring in practice, which results in the appearance of additional components (biases) in the cross-correlation function estimators determined based on quantized data.

Known in the literature is a relation for the cross-correlation function estimator bias determined based on signals quantized according to (5). It has the form [9]:

$$\begin{aligned} b\left[\tilde{R}^q_{xy}(k, M)\right] = & \frac{q_1}{2\pi}\sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \frac{(-1)^i}{i}\partial_{v_2}\Phi_{xy}(v_1, v_2)\Big|_{(v_1,v_2)=(0,0)} \\ & + \frac{q_2}{2\pi}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{(-1)^l}{l}\partial_{v_1}\Phi_{xy}(v_1, v_2)\Big|_{(v_1,v_2)=(0,0)} \\ & - \frac{q_1 q_2}{4\pi^2}\sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{(-1)^{i+l}}{il}\Phi_{xy}\left(-\frac{2\pi}{q_1}i, -\frac{2\pi}{q_2}l\right) \end{aligned} \quad (7)$$

where:

$$\partial_{v_1}\Phi_{xy}(v_1, v_2) = \frac{\partial}{\partial v_1}\Phi_{xy}\left(v_1, v_2 - \frac{2\pi}{q_2}l\right) \quad (7a)$$

$$\partial_{v_2}\Phi_{xy}(v_1, v_2) = \frac{\partial}{\partial v_2}\Phi_{xy}\left(v_1 - \frac{2\pi}{q_1}i, v_2\right) \quad (7b)$$

In practice, bias (7) is non-zero.

Bias of the estimator obtained based on randomly quantized signals

In the measurement technique, randomization of quantization is aimed to eliminate or reduce the estimator bias and can be realized by means of a dither signal or by inputting an appropriate random signal into the quantizer. In Figs 1 and 2 block diagrams of such realizations are shown.

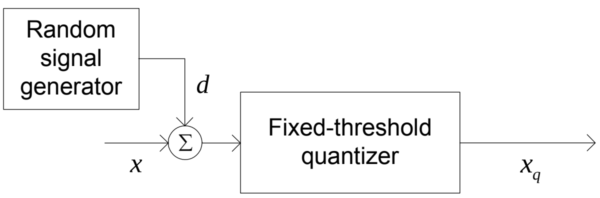


Fig.1. Quantizer randomized by non-substractive dither d

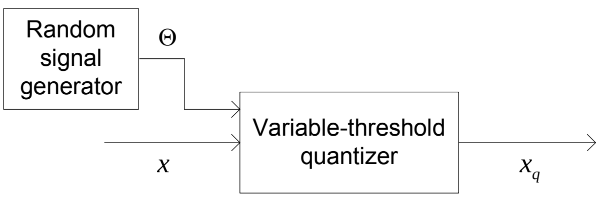


Fig.2. Quantizer randomized by signal Θ

Quantization with dither can be performed in accordance with the relation:

$$x_q = q \cdot \text{ent}\left(\frac{x+d}{q} + \frac{1}{2}\right) \quad (8)$$

where d is a random variable assuming the values of the dither signal. Such a signal is stationary and independent of the quantized signal [3]. The random variable d has a symmetric probability density function and a zero mean.

In measurement procedures, processing with dither must be accompanied by averaging [3]. In algorithm (2), such averaging takes place. The cross-correlation function is determined in a two-channel configuration and requires the application of two dither signals. If independent random variables assuming the values of dither signals are denoted by d_1 and d_2, then the systematic error (bias) of the estimator assumes the form [9][2]:

$$\begin{aligned} b\left[\tilde{R}^d_{xy}(k, M)\right] = & \frac{q_1}{2\pi}\sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \frac{(-1)^i}{i}\Phi_{d1}\left(\frac{2\pi}{q_1}i\right)\partial_{v_2}\Phi_{xy}(v_1, v_2)\Big|_{(v_1,v_2)=(0,0)} \\ & + \frac{q_2}{2\pi}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{(-1)^l}{l}\Phi_{d2}\left(\frac{2\pi}{q_2}l\right)\partial_{v_1}\Phi_{xy}(v_1, v_2)\Big|_{(v_1,v_2)=(0,0)} \\ & - \frac{q_1 q_2}{4\pi^2}\sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{(-1)^{i+l}}{il}\Phi^d_{xy}(v_1, v_2)\Big|_{(v_1,v_2)=\left(\frac{2\pi}{q_1}i, \frac{2\pi}{q_2}l\right)} \end{aligned} \quad (9)$$

where:

$$\Phi^d_{xy}(v_1, v_2) = \Phi_{d1}(v_1)\Phi_{d2}(v_2)\Phi_{xy}(-v_1, -v_2) \quad (10)$$

It follows from formula (9) that the bias equals zero when the characteristic functions of the random variables d_1 and d_2 satisfy the conditions:

$$\Phi_{d1}\left(\frac{2\pi}{q_1}i\right) = 0 \text{ for } i = \pm 1, \pm 2, \ldots \quad (11a)$$

$$\Phi_{d2}\left(\frac{2\pi}{q_2}l\right) = 0 \text{ for } l = \pm 1, \pm 2, \ldots \quad (11b)$$

or

$$\Phi_{d1}(v_1) = 0 \text{ for } |v_1| > \frac{2\pi}{q_1} \quad (12a)$$

$$\Phi_{d2}(v_2) = 0 \text{ for } |v_2| > \frac{2\pi}{q_2} \quad (12b)$$

Conditions (11a, b) are necessary and sufficient, whereas conditions (12a, b) are sufficient. The quantization shown in Fig. 2 is realized based on the relation:

$$x_q = q \cdot \text{ent}\left(\frac{x}{q} + 1 - \Theta\right) \quad (13)$$

where Θ is a random variable, e.g. with uniform distribution over the interval <0,1> [1, 2].

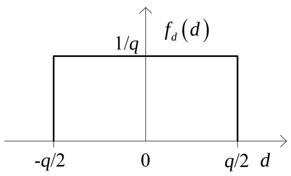
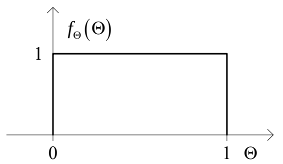
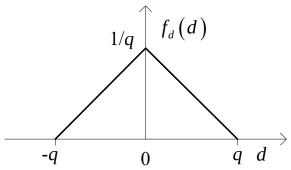
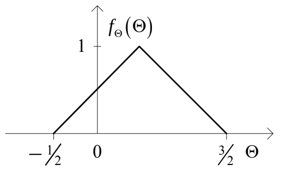
It can be shown that the mathematical operations described by formulae (8) and (13) are equivalent when:

$$\Theta = \frac{1}{2} - \frac{d}{q} \quad (14)$$

[2] In relation to [9], the formula has been simplified by using the property: $\Phi_{d1}(-v)=\Phi_{d1}(v)$ and $\Phi_{d2}(-v)=\Phi_{d2}(v)$.

Table 1 compiles the statistics of random variables d and Θ significant in modeling quantization operations.

Table 1. Properties of random variables d and Θ

Variable d	Variable Θ=0.5-d/q
with uniform density	
	
$f_d(d)=\begin{cases}\frac{1}{q} & \text{dla} \quad d\in\left\langle -\frac{q}{2},\frac{q}{2}\right\rangle \\ 0 & \text{dla} \quad d\notin\left\langle -\frac{q}{2},\frac{q}{2}\right\rangle\end{cases}$	$f_\Theta(\Theta)=\begin{cases}1 & \text{dla} \quad \Theta\in<0,1> \\ 0 & \text{dla} \quad \Theta\notin<0,1>\end{cases}$
$\Phi_d(v)=\mathrm{sinc}\left(\frac{q}{2}v\right)$	$\Phi_\Theta(v)=\frac{e^{jv}-1}{jv}$
$\Phi_d\left(\frac{2\pi}{q}i\right)=0,\ i=\pm1, \pm2,\ldots$	$\Phi_\Theta(2\pi i)=0,\ i=\pm1, \pm2,\ldots$
with triangular density	
	
$f_d(d)=\begin{cases}\frac{1}{q}\left(1-\frac{\lvert d\rvert}{q}\right) & \text{dla } d\in<-q,q> \\ 0 & \text{dla } d\notin<-q,q>\end{cases}$	$f_\Theta(\Theta)=\begin{cases}1-\frac{1}{2}\lvert 2\Theta-1\rvert & \text{dla} \quad \Theta\in\left\langle -\frac{1}{2},\frac{3}{2}\right\rangle \\ 0 & \text{dla} \quad \Theta\notin\left\langle -\frac{1}{2},\frac{3}{2}\right\rangle\end{cases}$
$\Phi_d(v)=\mathrm{sinc}^2\left(\frac{q}{2}v\right)$	$\Phi_\Theta(v)=\frac{4}{v^2}\sin^2\left(\frac{1}{2}v\right)e^{j\frac{1}{2}v}$
$\Phi_d\left(\frac{2\pi}{q}i\right)=0,\ i=\pm1, \pm2,\ldots$	$\Phi_\Theta(2\pi i)=0,\ i=\pm1, \pm2,\ldots$

Making use of the properties of the characteristic function [10], as well as of the fact that:

(15) $$d=-q\Theta+\frac{1}{2}q$$

we can obtain the expressions:

(16) $$\Phi_d(v)=e^{j\frac{1}{2}qv}\Phi_\Theta(qv)$$

(17) $$\Phi_d\left(\frac{2\pi}{q}i\right)=(-1)^i\,\Phi_\Theta(2\pi i),\ i=\pm1, \pm2,\ldots$$

By virtue of (9) and (17), we can formulate a relation for the cross-correlation function bias obtained in a configuration with quantizers randomized by signals Θ_1 and Θ_2. It assumes the form:

(18) $$b\left[\tilde{R}_{xy}^{\Theta}(k,M)\right]=\frac{q_1}{2\pi}\sum_{\substack{i=-\infty\\ i\neq0}}^{\infty}\frac{1}{i}\Phi_{\Theta1}(2\pi i)\partial_{v_2}\Phi_{xy}(v_1,v_2)\Big|_{(v_1,v_2)=(0,0)}+\frac{q_2}{2\pi}\sum_{\substack{l=-\infty\\ l\neq0}}^{\infty}\frac{1}{l}\Phi_{\Theta2}(2\pi l)\partial_{v_1}\Phi_{xy}(v_1,v_2)\Big|_{(v_1,v_2)=(0,0)}-\frac{q_1q_2}{4\pi^2}\sum_{\substack{i=-\infty\\ i\neq0}}^{\infty}\sum_{\substack{l=-\infty\\ l\neq0}}^{\infty}\frac{1}{il}\Phi_{xy}^{\Theta}(v_1,v_2)\Big|_{(v_1,v_2)=\left(\frac{2\pi}{q_1}i,\frac{2\pi}{q_2}l\right)}$$

where:

(19) $$\Phi_{xy}^{\Theta}(v_1,v_2)=\Phi_{\Theta1}(q_1v_1)\Phi_{\Theta2}(q_2v_2)\Phi_{xy}(-v_1,-v_2)$$

Random signals with distributions corresponding to the random variable distributions from Table 1 make it possible to bring relations (9) and (18) to zero, which signifies a theoretical possibility of eliminating the estimator bias (systematic error) resulting from not fulfilling the condition expressed in formula (6) by the quantized signals.

It is possible to randomize flash converters by acting upon the reference voltage [1].

Conclusion

The influence of three ways of quantization on the bias (systematic error) of digital correlation function estimators has been analyzed. An original relation (18) for the bias of an estimator realized in a system with quantizers randomized by random signals has been derived. Two types of random signals (with uniform and triangular distributions), whose use makes it possible to (theoretically) bring the estimator bias to zero, have been analyzed.

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